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In this chapter, we shall learn about some basics of mathematical reasoning.
In mathematical language, there are two kinds of reasoning; one is inductive reasoning and another is deductive reasoning.
In this chapter, we shall study about some curves as circles, ellipses, parabolas and hyperbolas. These curves are known as conic sections or conics because they can be obtained as intersections of a plane with a double napped right circular cone.

CONIC SECTIONS

|TOPIC 1|

Sections of a Cone and Circle

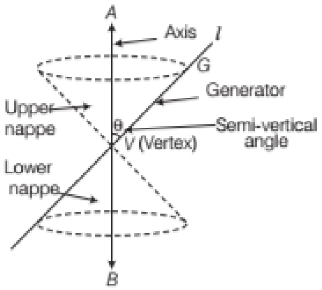
CHAPTER CHECKLIST
- Sections of a Cone and Circle
- Parabola
- Ellipse
- Hyperbola

RIGHT CIRCULAR CONE

The locus of a line passing through a fixed point say A and making a constant angle θ with a fixed line AB passing through the fixed point A is called **right circular cone.**

Here, the fixed point A is called *vertex*, the fixed line AB is called the *axis* and the constant angle θ is called the *semi-vertical angle* of the cone. Also, the moving line is called the *generator* of the cone.

Here, the fixed line *AB* is called *axis*, fixed point *V* is called *vertex* of the cone and the constant angle θ is called the *semi-vertical angle* of the cone. The rotated line *l* in all its positions is called a *generator* of the cone. The two parts of the generated surface separated by vertex *V* are called ***nappes***.



CONIC SECTIONS

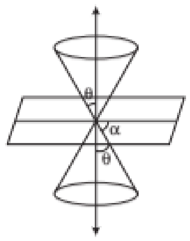
On intersecting a right circular cone by a plane in different positions, different sections so obtained are called conic sections or conics.

Section of a Doubled Napped Right Circular Cone by a Plane

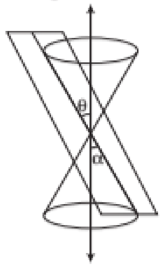
Consider a double napped right circular cone, having semi-vertical angle θ. Let α be the angle between the plane and the axis of cone.

Case I **When the plane passes through vertex of the cone**
If the plane passes through the vertex of the cone, then it cuts both nappies. According to the value of α, we get the following sections:

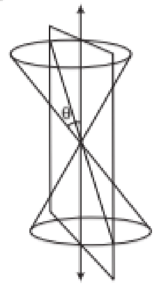
(i) When $\theta < \alpha \le 90°$, then the section is a **point**.



(ii) When $\theta = \alpha$, then the plane contains the generator of cone and the section is a **straight line**. It is a **degenerated case of a parabola.**

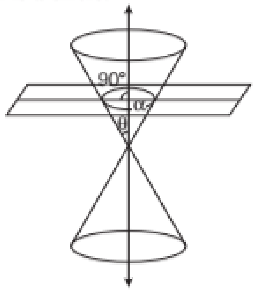


(iii) When $0 \le \alpha < \theta$, then section is a pair of straight lines. It is a **degenerated case of a hyperbola.**

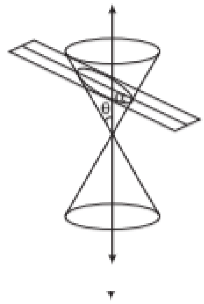


Case II **When the plane does not pass through vertex of the cone** If the plane does not pass through the vertex of the cone, then it cuts only one nappe. According to the value of α, we get the following sections:

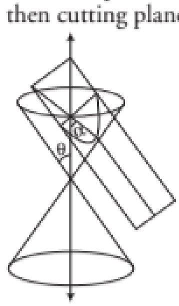
(i) When $\alpha = 90°$, then cutting plane section is a **circle**.



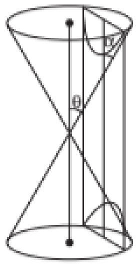
(ii) When $\theta < \alpha < 90°$, then cutting plane section is an **ellipse**.



(iii) When $\alpha = \theta$ i.e. plane is parallel to a generator, then cutting plane is **parabola.**



(iv) When $0 \le \alpha < \theta$, then the plane intersects both the nappies and section is a **hyperbola**.



Geometrical Definition of Conic Section

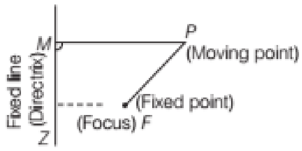
A conic is the locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point and a fixed line is a constant. Then,

(i) the fixed point is called **focus** and is denoted by F.

(ii) the fixed straight line ZM is called **directrix**.

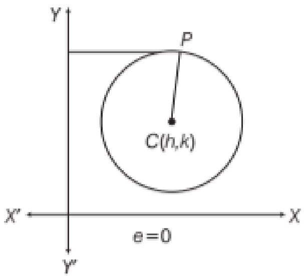
(iii) the constant ratio is called **eccentricity** and is denoted by e.

In the given figure, $\dfrac{PF}{PM} = e =$ constant

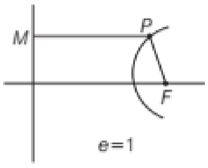


On the basis of the value of e, we get the following cases

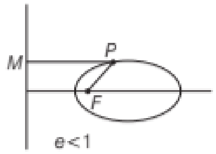
(a) **Circle** If eccentricity, $e = 0$, then the conic is called circle.



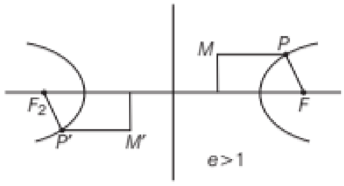
(b) **Parabola** If eccentricity, $e = 1$, then the conic is called parabola.



(c) **Ellipse** If eccentricity, $e < 1$, then the conic is called ellipse.



(d) **Hyperbola** If eccentricity, $e > 1$, then the conic is called hyperbola.



(iv) The straight line passing through the focus and perpendicular to the directrix, is called the **axis** of the conic.

(v) The point of intersection of the cone and its axis, is called **vertex** of the conic.

(vi) A line perpendicular to the axis of the conic and passing through its focus is called **latusrectum**.

(vii) The point which bisects every chord of the conic passing through it, is called the **centre** of the conic.

CIRCLE

A circle is the set of all points in a plane that are equidistant from a fixed point in that plane.

Or

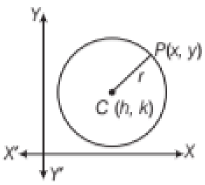
A circle is defined as the locus of a point in a plane, which moves in a plane such that its distance from a fixed point in that plane is always constant.

Centre The fixed point C is called the centre of the circle.

Radius The constant distance (CP) from the centre (C) to a point on the circle, is called radius (r).

Standard Equation of a Circle

Let $C(h, k)$ be the *centre* of the circle, $P(x, y)$ be any point on the circumference of the circle and r be the *radius* of the circle.



Then, equation of circle in standard form is

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

This equation is also known as central form of the equation of a circle

Proof From the figure, $CP=r$

$\Rightarrow \sqrt{(x-h)^2+(y-k)^2}=r$

[by distance formula, distance $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$]

$\Rightarrow (x-h)^2+(y-k)^2=r^2$ [squaring both sides] ...(i)

If $h=0$ and $k=0$ i.e. centre of the circle is at origin $(h,k)=(0,0)$, then from Eq. (i), we get

$$\boxed{x^2+y^2=r^2}$$

EXAMPLE |1| Find the equation of the circle with

(i) centre = (2, 3) and radius = 5

(ii) centre = $(a\cos\theta, a\sin\theta)$ and radius = a

(iii) centre = $(-a,-b)$ and radius = $\sqrt{a^2-b^2}$ **[NCERT]**

(iv) centre = $\left(\frac{1}{2},\frac{1}{4}\right)$ and radius = $\frac{1}{12}$ units **[NCERT]**

Sol. (i) Given centre is (2, 3).

$\therefore$ $h=2, k=3$ and radius $(r)=5$

On putting these values in equation of circle $(x-h)^2+(y-k)^2=r^2$, we get

$$(x-2)^2+(y-3)^2=5^2$$

$\Rightarrow x^2+4-4x+y^2+9-6y=25$

$[\because (A-B)^2=A^2+B^2-2AB]$

$\Rightarrow x^2+y^2-4x-6y+13=25$

$\Rightarrow x^2+y^2-4x-6y+13-25=0$

$\therefore x^2+y^2-4x-6y-12=0$

which is the required equation of circle.

(ii) Given centre is $(a\cos\theta, a\sin\theta)$.

$\therefore h=a\cos\theta, k=a\sin\theta$ and radius $(r)=a$

On putting these values in equation of circle $(x-h)^2+(y-k)^2=r^2$, we get

$$(x-a\cos\theta)^2+(y-a\sin\theta)^2=a^2$$

$\Rightarrow x^2+a^2\cos^2\theta-2ax\cos\theta+y^2+a^2\sin^2\theta-2ay\sin\theta=a^2$

$\Rightarrow x^2+y^2-2ax\cos\theta-2ay\sin\theta+a^2(\sin^2\theta+\cos^2\theta)=a^2$

$\Rightarrow x^2+y^2-2a(x\cos\theta+y\sin\theta)+a^2=a^2$

$[\because \sin^2\theta+\cos^2\theta=1]$

$\therefore x^2+y^2-2a(x\cos\theta+y\sin\theta)=0$

which is the required equation of circle.

(iii) Given centre is $(-a,-b)$.

$\therefore h=-a, k=-b$ and radius $(r)=\sqrt{a^2-b^2}$

On putting these values in equation of circle $(x-h)^2+(y-k)^2=r^2$, we get

$$[x-(-a)]^2+[y-(-b)]^2=(\sqrt{a^2-b^2})^2$$

$\Rightarrow (x+a)^2+(y+b)^2=a^2-b^2$

$\Rightarrow x^2+a^2+2ax+y^2+b^2+2by=a^2-b^2$

$[\because (A+B)^2=A^2+2AB+B^2]$

$\Rightarrow x^2+y^2+2ax+2by+a^2+b^2=a^2-b^2$

$\therefore x^2+y^2+2ax+2by+2b^2=0$

which is the required equation of circle.

(iv) Given centre is $\left(\frac{1}{2},\frac{1}{4}\right)$.

$\therefore h=\frac{1}{2}, k=\frac{1}{4}$ and radius $(r)=\frac{1}{12}$ unit

On putting these values in equation of circle $(x-h)^2+(y-k)^2=r^2$, we get

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{4}\right)^2=\left(\frac{1}{12}\right)^2$$

$\Rightarrow x^2+\frac{1}{4}-x+y^2+\frac{1}{16}-\frac{y}{2}=\frac{1}{144}$

$\Rightarrow x^2+y^2-x-\frac{y}{2}+\frac{1}{4}+\frac{1}{16}-\frac{1}{144}=0$

$\Rightarrow x^2+y^2-x-\frac{y}{2}+\frac{11}{36}=0$

$\Rightarrow 36x^2+36y^2-36x-18y+11=0$

[multiplying both sides by 36]

which is the required equation of circle.

EXAMPLE |2| Find the equation of circle whose centre is (1, 2) and which passes through the point (4, 6).

Sol. Coordinates of centre of given circle is (1, 2) and it passes through the point (4, 6).

Then, radius of the circle is equal to the distance from the centre to a point on a circle.

$\therefore$ Radius of circle $=\sqrt{(1-4)^2+(2-6)^2}=\sqrt{9+16}=5$

[by distance formula, distance $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$]

Hence, the required equation of the circle is

$$(x-1)^2+(y-2)^2=(5)^2$$

$\Rightarrow x^2+y^2-2x-4y-20=0$

Equation of Circle in Special Cases

We know that, the standard equation of the circle with centre at (h,k) and radius equal to r, is $(x-h)^2+(y-k)^2=r^2$.

It can be change according to the different conditions, *which are given below*

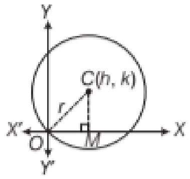
|CASE I|

WHEN THE CIRCLE PASSES THROUGH THE ORIGIN

The circle passes through origin $(0,0)$.

$\therefore$ Radius $= r =$ Distance between points O and C

$$= \sqrt{h^2 + k^2}$$



Hence, the equation of a circle passing through the origin is

$$(x-h)^2 + (y-k)^2 = h^2 + k^2$$

i.e. $$\boxed{x^2 + y^2 - 2hx - 2ky = 0}$$

EXAMPLE |3| Find the equation of the circle whose centre is (a, b) and passes through the origin.

Sol. We know that, circle passes through the origin, so radius of circle will be equal to the distance between point (a, b) and origin.

$\therefore$ Radius of circle = Distance between points $(0, 0)$ and (a, b)

$$= \sqrt{(0-a)^2 + (0-b)^2} = \sqrt{a^2 + b^2} \quad \text{[by distance formula]}$$

$\because$ Centre $= (h, k) = (a, b)$

On putting these values in equation of circle

$(x-h)^2 + (y-k)^2 = r^2$, we get

$$(x-a)^2 + (y-b)^2 = (\sqrt{a^2+b^2})^2$$

$$\Rightarrow x^2 + a^2 - 2ax + y^2 + b^2 - 2by = a^2 + b^2$$

$$[\because (A-B)^2 = A^2 - 2AB + B^2]$$

$$\Rightarrow \quad x^2 + y^2 - 2ax - 2by = 0$$

which is the required equation of circle.

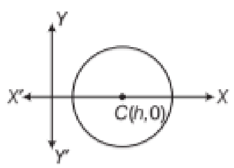
|CASE II|

WHEN THE CENTRE LIES ON *X*-AXIS OR *Y*-AXIS

If centre lies on X-axis, then $k = 0$.

$\therefore$ Equation of circle with centre $C(h, 0)$ is

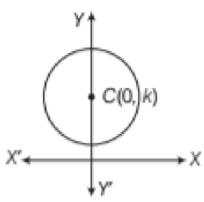
$$\boxed{(x-h)^2 + y^2 = r^2}$$



If centre lies on Y-axis, then $h = 0$.

$\therefore$ Equation of circle with centre $C(0, k)$ is

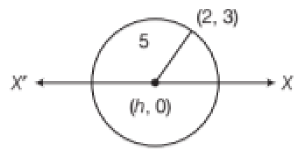
$$\boxed{x^2 + (y-k)^2 = r^2}$$



EXAMPLE |4| If the circle passes through the point (2,3), then find the equation of the circle whose radius is 5 units and centre lies on X-axis.

Sol. Given, radius, $r = 5$ units, point lies on circle $= (2, 3)$ and centre of circle is on X-axis.

$\therefore$ Its y-coordinate will be zero.



Let centre of circle $= (h, 0)$

Now, distance from the centre to a point on the circle = Radius of circle

$$\therefore \quad \sqrt{(h-2)^2 + (0-3)^2} = 5 \quad \text{[by distance formula]}$$

$$\Rightarrow \quad \sqrt{h^2 + 4 - 4h + 9} = 5 \quad [\because (A-B)^2 = A^2 + B^2 - 2AB]$$

$$\Rightarrow \quad h^2 + 4 - 4h + 9 = 25 \quad \text{[squaring on both sides]}$$

$$\Rightarrow \quad h^2 - 4h + 13 - 25 = 0 \Rightarrow \quad h^2 - 4h - 12 = 0$$

$$\Rightarrow \quad h^2 - 6h + 2h - 12 = 0 \Rightarrow h(h-6) + 2(h-6) = 0$$

$$\Rightarrow \quad (h-6)(h+2) = 0$$

$$\Rightarrow \quad h - 6 = 0 \text{ or } h + 2 = 0 \Rightarrow h = 6 \text{ or } h = -2$$

So, the centre of circle is $(6, 0)$ or $(-2, 0)$.

When centre $(h, k) = (6, 0)$ and radius $(r) = 5$, then the equation of circle is

$$(x-6)^2 + (y-0)^2 = (5)^2 \quad [\because (x-h)^2 + (y-k)^2 = r^2]$$

$$\Rightarrow \quad x^2 + 36 - 12x + y^2 = 25 \quad [\because (a-b)^2 = a^2 - 2ab + b^2]$$

$$\Rightarrow \quad x^2 + y^2 - 12x + 11 = 0$$

When centre $(h, k) = (-2, 0)$ and radius $(r) = 5$, then the equation of circle is

$$(x+2)^2 + (y-0)^2 = (5)^2 \quad [\because (x-h)^2 + (y-k)^2 = r^2]$$

$$\Rightarrow \quad x^2 + 4 + 4x + y^2 = 25$$

$$\Rightarrow x^2 + y^2 + 4x + 4 - 25 = 0$$

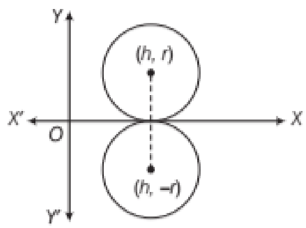
$$\Rightarrow \quad x^2 + y^2 + 4x - 21 = 0$$

Hence the equations of circle are $x^2 + y^2 - 12x + 11 = 0$ and $x^2 + y^2 + 4x - 21 = 0$.

|CASE III|

WHEN CIRCLES TOUCHES *X*-AXIS

Since, the circles touches X-axis, so $|k| = r$. In this case, circles may lie on upper of X-axis or on the lower of X-axis.



So, if r is the radius of a circle touching X-axis, then its centre is (h, r) or $(h, -r)$.

Therefore, the equations of such circles are

$$\boxed{(x-h)^2+(y\mp r)^2=r^2}$$

EXAMPLE |5| Find the equation of circle whose centre is (1, 2) and touches X-axis.

Sol. Given, centre $(h, k) = (1, 2)$

and circle touches on X-axis.

$\therefore$ Radius (r) = y-coordinate of centre = 2

So, equation of circle is

$$(x-1)^2+(y-2)^2=2^2 \quad [\because (x-h)^2+(y-k)^2=r^2]$$

$$\Rightarrow \quad x^2-2x+1+y^2-4y+4=4$$

$$[\because (a-b)^2=a^2+b^2-2ab]$$

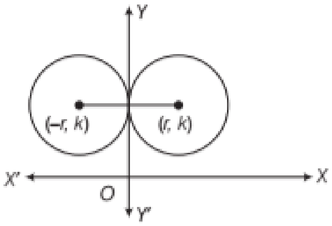
$$\Rightarrow \quad x^2+y^2-2x-4y+1=0$$

which is the required equation of circle.

|CASE IV|

WHEN CIRCLES TOUCHES Y-AXIS

Since, the circles touches Y-axis, so $|h| = r$.



In this case, circles may lie on the right of Y-axis or on the left of Y-axis. So, if r is the radius of a circle touching Y-axis, then its centre is (r, k) or $(-r, k)$.

Therefore, the equation of such circles are

$$\boxed{(x\mp r)^2+(y-k)^2=r^2}$$

EXAMPLE |6| Find the equation of circle whose centre is (2, 0) and touches Y-axis.

Sol. Given, centre $(h, k) = (2, 0)$ and circle touches Y-axis.

$\therefore$ Radius (r) = x-coordinate of centre = 2

So, the equation of circle is

$$(x-2)^2+(y-0)^2=2^2 \quad [\because (x-h)^2+(y-k)^2=r^2]$$

$$\Rightarrow \quad x^2+4-4x+y^2=4 \quad [\because (A-B)^2=A^2+B^2-2AB]$$

$$\Rightarrow \quad x^2+y^2-4x+4=4$$

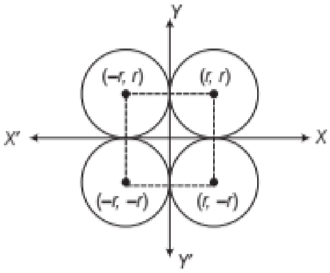
$$\therefore \quad x^2+y^2-4x=0$$

which is the required equation of circle.

|CASE V|

WHEN CIRCLES TOUCHES BOTH THE COORDINATE AXES

Since, the circles touches both the coordinate axes, so $|h| = |k| = r$.



In this case, circles touching both the coordinate axes may lie in any of the four quadrants. If r is the radius of such a circle, then its centre may be $(r, r), (-r, r), (-r, -r)$ and $(r, -r)$.

Then, the equations of such circles are

$$\boxed{(x\mp r)^2+(y\mp r)^2=r^2}$$

EXAMPLE |7| Find the equation of the circle which touches both the axes and whose radius is 5.

Sol. Given, radius = 5 and circle touches both the axes.

So, circle may lies in anyone of the four quadrants.

$\therefore$ Centre of circle is $(\pm 5, \pm 5)$.

Hence, the equation of required circle is

$$(x\mp 5)^2+(y\mp 5)^2=(5)^2$$

$$\Rightarrow \quad x^2+25\mp 10x+y^2+25\mp 10y=25$$

$$\Rightarrow \quad x^2+y^2\mp 10x\mp 10y+25=0$$

General Equation of a Circle

We know that the equation of the circle having centre (h, k) and radius r is $\quad (x-h)^2+(y-k)^2=r^2$

$$\Rightarrow \quad x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$$

which is of the form

$$x^2+y^2+2gx+2fy+c=0$$

where, $g=-h, \quad f=-k$

and $\quad c=h^2+k^2-r^2$

The above equation of a circle is called the general equation of a circle with centre $(-g, -f)$

and radius, $r=\sqrt{h^2+k^2-c}$

or $\quad r=\sqrt{g^2+f^2-c}$.

Different Types of Questions Based on Circle

|TYPE I|

TO FIND CENTRE AND RADIUS WHEN STANDARD EQUATION OF CIRCLE IS GIVEN

EXAMPLE |8| Find the centre and radius of each of the following circle.

(i) $x^2 + (y+2)^2 = 9$ (ii) $x^2 + y^2 + 6x - 4y + 4 = 0$

Sol. (i) Given equation of circle is

$$x^2 + (y+2)^2 = 9 \Rightarrow (x-0)^2 + \{y-(-2)\}^2 = 3^2$$

On comparing with $(x-h)^2 + (y-k)^2 = r^2$, we get

$$h = 0, k = -2 \text{ and } r = 3$$

Hence, centre of circle $= (0, -2)$ and radius $= 3$ units.

(ii) Given equation of circle is

$$x^2 + y^2 + 6x - 4y + 4 = 0$$

$$\Rightarrow (x^2 + 6x) + (y^2 - 4y) = -4$$

$$\Rightarrow (x^2 + 6x + 9 - 9) + (y^2 - 4y + 4 - 4) = -4$$

$$\Rightarrow (x^2 + 6x + 9) + (y^2 - 4y + 4) = -4 + 4 + 9$$

$$\Rightarrow (x+3)^2 + (y-2)^2 = 9$$

$$\Rightarrow \{x-(-3)\}^2 + (y-2)^2 = 3^2$$

On comparing with $(x-h)^2 + (y-k)^2 = r^2$, we get

$$h = -3, k = 2 \text{ and } r = 3$$

Hence, centre of circle $= (-3, 2)$ and radius $= 3$.

EXAMPLE |9| Find the centre and radius of each of the following circle.

(i) $3x^2 + 3y^2 = 27$ (ii) $x^2 + y^2 - 6x + 5y - 8 = 0$

Sol. (i) Given circle is $3x^2 + 3y^2 = 27$.

or $x^2 + y^2 = 9$ [dividing both sides by 3]

$$\Rightarrow (x-0)^2 + (y-0)^2 = 9$$

$$\Rightarrow (x-0)^2 + (y-0)^2 = 3^2 \quad \text{...(i)}$$

On comparing Eq. (i) with standard form of circle i.e. $(x-h)^2 + (y-k)^2 = r^2$, we get

Centre $(h, k) = (0, 0)$ and radius $(r) = 3$

(ii) Given equation of circle is

$$x^2 + y^2 - 6x + 5y - 8 = 0$$

$$\Rightarrow (x^2 - 6x) + (y^2 + 5y) = 8$$

$$\Rightarrow (x^2 - 6x + 3^2) + \left[y^2 + 5y + \left(\frac{5}{2}\right)^2\right] = 3^2 + \left(\frac{5}{2}\right)^2 + 8$$

$$\Rightarrow (x-3)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{93}{4} \quad \text{...(i)}$$

On comparing Eq. (i) with standard form of circle i.e. $(x-h)^2 + (y-k)^2 = r^2$, we get

$$h = 3, k = -\frac{5}{2} \text{ and } r = \frac{\sqrt{93}}{2}$$

|TYPE II|

TO FIND THE EQUATION OF A CIRCLE SATISFYING THE GIVEN CONDITIONS

EXAMPLE |10| Find the equation of the circle passing through the point (2, 4) and having its centre at the intersection of the lines $x - y = 4$ and $2x + 3y + 7 = 0$.

Sol. Given equation of lines are $x - y = 4$ and $2x + 3y + 7 = 0$.

On solving these equations, we get $x = 1$ and $y = -3$

Also, the point of intersection of the given lines is centre.

$\therefore$ Coordinates of centre is $(1, -3)$.

Now, the circle passes through the point (2, 4).

$\therefore$ Radius of circle $= \sqrt{(1-2)^2 + (-3-4)^2} = \sqrt{1+49} = \sqrt{50}$

Hence, the required equation of circle whose centre is $(1, -3)$ and radius is $\sqrt{50}$, is

$$(x-1)^2 + (y+3)^2 = (\sqrt{50})^2$$

$$\Rightarrow x^2 + 1 - 2x + y^2 + 9 + 6y = 50$$

$$\Rightarrow x^2 + y^2 - 2x + 6y - 40 = 0$$

EXAMPLE |11| Find the equation of the circle, whose centre is $(2, -3)$ and passing through the intersection of the lines $3x - 2y = 1$ and $4x + y = 27$.

Sol. Let $P(x, y)$ be the point of intersection of the lines,

$$3x - 2y = 1 \quad \text{...(i)}$$

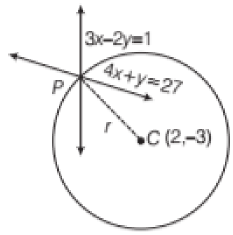
and

$$4x + y = 27 \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$x = 5 \text{ and } y = 7$$

So, the coordinates of P are $(5, 7)$.



Given, coordinates of centre C are $(2, -3)$.

$$\therefore CP = \text{Radius}$$

$$\Rightarrow \sqrt{(5-2)^2 + (7+3)^2} = r$$

$$\Rightarrow \sqrt{9 + 100} = r$$

$$\Rightarrow r = \sqrt{109}$$

Hence, the equation of required circle is

$$(x-2)^2 + (y+3)^2 = (\sqrt{109})^2$$

$$\Rightarrow x^2 - 4x + 4 + y^2 + 6y + 9 = 109$$

$$\Rightarrow x^2 + y^2 - 4x + 6y - 96 = 0$$

EXAMPLE |12| Find the equation of a circle whose diameter are $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$ and area is 154 sq units.

Sol. Given equations of diameter of circles are

$$2x - 3y + 12 = 0 \text{ and } x + 4y - 5 = 0$$

On solving these equations, we get $x = -3$ and $y = 2$

We know that centre is the point of intersection of diameter.

$\therefore$ Coordinates of centres are $(-3, 2)$.

Let r be the radius of circle.

Then, area of circle $= 154 = \pi r^2$

$$\Rightarrow \quad 154 = \frac{22}{7} \times r^2 \Rightarrow r^2 = \frac{154 \times 7}{22} = 49$$

$$\therefore \quad r = 7 \qquad [\because r > 0]$$

Hence, the equation of required circle is

$$(x+3)^2 + (y-2)^2 = (7)^2$$

$$\Rightarrow \quad x^2 + 6x + 9 + y^2 - 4y + 4 = 49$$

$$\Rightarrow \quad x^2 + y^2 + 6x - 4y + 13 - 49 = 0$$

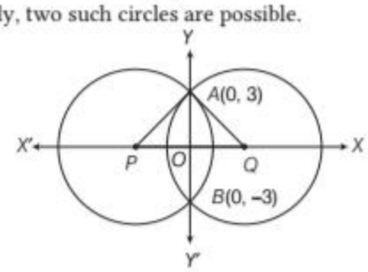
$$\Rightarrow \quad x^2 + y^2 + 6x - 4y - 36 = 0$$

EXAMPLE |13| Find the equation of the circle passing through two points on Y-axis at distances 3 from the origin and having radius 5.

Sol. Let coordinates of points on Y-axis be $A \equiv (0, 3)$ and $B \equiv (0, -3)$, respectively.

Given circle passes through points A and B, therefore its centre will be on the perpendicular bisector of AB.

Clearly, two such circles are possible.



Let points P and Q be the centres of such circles.

Given, $AP = AQ = 5$, $OA = 3 \Rightarrow OP = OQ = \sqrt{5^2 - 3^2} = 4$

Then, centres $P = (-4, 0)$ and $Q = (4, 0)$

Thus, centres of the required circles are $P(-4, 0)$ and $Q(4, 0)$ and their radii are 5.

$\therefore$ Equation of the required circles will be

$$(x+4)^2 + y^2 = 5^2 \text{ i.e. } x^2 + y^2 + 8x - 9 = 0$$

$$\text{and } (x-4)^2 + y^2 = 5^2 \text{ i.e. } x^2 + y^2 - 8x - 9 = 0$$

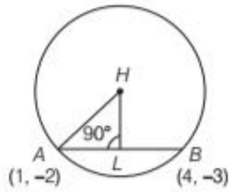
EXAMPLE |14| Find the equation of the circle passing through the points $(1, -2)$ and $(4, -3)$ and whose centre lies on the line $3x + 4y = 7$.

Sol. Let points $A \equiv (1, -2)$ and $B \equiv (4, -3)$

Given line is $3x + 4y = 7$ ...(i)

Again, let L is the mid-point of AB.

$$\therefore \text{ Mid-piont } L \equiv \left(\frac{1+4}{2}, \frac{-2+(-3)}{2}\right) = \left(\frac{5}{2}, -\frac{5}{2}\right)$$



$$\text{Slope of } AB = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-2+3}{1-4} = -\frac{1}{3}$$

$$\therefore \quad \text{Slope of } HL = 3 \qquad [\because HL \perp AB]$$

Then, equation of HL will be $y + \frac{5}{2} = 3\left(x - \frac{5}{2}\right)$

$$\Rightarrow \quad 2y + 5 = 6x - 15$$

$$\Rightarrow \quad 6x - 2y = 20$$

$$\therefore \quad 3x - y = 10 \qquad \text{...(ii)}$$

Centre H of the circle lies on lines (i) and (ii), therefore it is the point of intersection of lines (i) and (ii).

On multiplying Eq. (ii) by 4 and adding it to Eq. (i), we get

$$15x = 47 \Rightarrow x = \frac{47}{15}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$5y = -3 \Rightarrow y = -\frac{3}{5}$$

$$\therefore \quad \text{Centre } H \equiv \left(\frac{47}{15}, -\frac{3}{5}\right)$$

Now, radius of the circle $= HA = \sqrt{\left(\frac{47}{15} - 1\right)^2 + \left(-\frac{3}{5} + 2\right)^2}$

$$= \sqrt{\left(\frac{32}{15}\right)^2 + \left(\frac{7}{5}\right)^2} = \sqrt{\frac{32^2 + 21^2}{15^2}} = \frac{\sqrt{1465}}{15}$$

Hence, the equation of required circle is

$$\left(x - \frac{47}{15}\right)^2 + \left(y + \frac{3}{5}\right)^2 = \frac{1465}{15^2}$$

$$\Rightarrow \quad (15x - 47)^2 + 9(5y+3)^2 = 1465$$

$$\Rightarrow \quad 225x^2 + 2209 - 1410x + 225y^2 + 81 + 270y = 1465$$

$$\Rightarrow \quad 225x^2 + 225y^2 - 1410x + 270y + 825 = 0$$

$$\Rightarrow \quad 15x^2 + 15y^2 - 94x + 18y + 55 = 0$$

EXAMPLE |15| Find the equation of a circle passing through the point $(7, 3)$ having radius 3 units and whose centre lies on the line $y = x - 1$.

Sol. Let (h, k) be the centre of circle.

$\therefore$ (h, k) lies on the line $y = x - 1$, then $k = h - 1$

Therefore, the equation of circle is

$$(x-h)^2 + [y - (h-1)]^2 = (3)^2 \qquad \text{...(i)}$$

Since, the circle passes through the point (7, 3).

$\therefore \quad (7-h)^2+(3-h+1)^2=9$

$\Rightarrow \quad (7-h)^2+(4-h)^2=9$

$\Rightarrow 49-14h+h^2+16-8h+h^2=9 \Rightarrow h^2-11h+28=0$

$\Rightarrow \quad (h-7)(h-4)=0 \Rightarrow h=7$ or $h=4$

When $h=7$, then $k=7-1=6$ and when $h=4$, then $k=4-1=3$

Hence, the required equation of circles are

$(x-7)^2+(y-6)^2=3^2$ or $(x-4)^2+(y-3)^2=3^2$

$\Rightarrow \quad x^2+y^2-14x-12y+76=0$

or $\quad x^2+y^2-8x-6y+16=0$

EXAMPLE |16| Find the equation of a circle which touches both the axes and the line $3x-4y+8=0$ and lies in third quadrant.

Sol. Let a be the radius of the circle, then $(-a,-a)$ will be centre of the circle. [$\because$ circle lies in 3rd quadrant]

Also, the line $3x-4y+8=0$ touches the circle.

Therefore, the perpendicular distance from centre to the given line is the radius of the circle.

$$\therefore a=\left|\frac{3(-a)-4(-a)+8}{\sqrt{3^2+4^2}}\right| \Rightarrow a=\left|\frac{-3a+4a+8}{5}\right|$$

$\Rightarrow \quad 5a=a+8 \Rightarrow a=2$

Hence, the equation of the required circle is

$(x+2)^2+(y+2)^2=2^2$

$\Rightarrow \quad x^2+4x+4+y^2+4y+4=4$

$\Rightarrow \quad x^2+y^2+4x+4y+8-4=0$

$\therefore \quad x^2+y^2+4x+4y+4=0$

EXAMPLE |17| Find the equation of circle whose centre is $(3,-1)$ and which cuts off a chord of length 6 units on the line $2x-5y+18=0$.

Sol. Given, centre of circle, $O\equiv(3,-1)$.

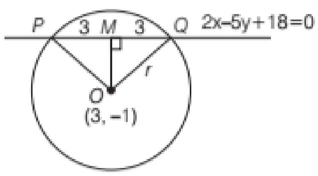
Let OM be the perpendicular distance from O to the line $2x-5y+18=0$.

Then, $\quad OM=\left|\frac{2(3)-5(-1)+18}{\sqrt{(2)^2+(5)^2}}\right|=\left|\frac{29}{\sqrt{29}}\right|=\sqrt{29}$

In ΔOMP, $OP^2=OM^2+PM^2$ [by Pythagoras theorem]

$\Rightarrow \quad OP^2=(\sqrt{29})^2+(3)^2 \quad \left[\because PM=\frac{1}{2}PQ=\frac{1}{2}\times 6=3\right]$

$\Rightarrow \quad OP^2=29+9=38$



Hence, the equation of required circle is

$(x-3)^2+(y+1)^2=38$

$\Rightarrow \quad x^2-6x+9+y^2+2y+1=38$

$\Rightarrow \quad x^2+y^2-6x+2y+10-38=0$

$\Rightarrow \quad x^2+y^2-6x+2y-28=0$

EXAMPLE |18| A circle of radius 2 lies in the first quadrant and touches both the axes of coordinates. Find the equation of the circle with centre at (6, 5) and touching the above circle externally.

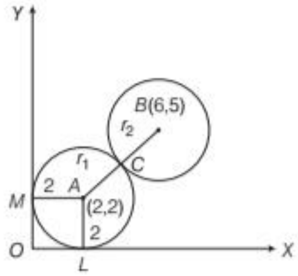
Sol. Given, $\quad AM=AL=AC=2$

and $\quad$ points $A\equiv(2,2)$, $B\equiv(6,5)$

$\therefore \quad$ Distance $AB=\sqrt{(2-6)^2+(2-5)^2}$

$=\sqrt{(4)^2+(3)^2}=5$

Then, distance $BC=AB-AC=5-2=3$



Hence, the equation of required circle whose centre is (6, 5) and radius is 3, will be

$(x-6)^2+(y-5)^2=3^2$

$\Rightarrow \quad x^2+y^2-12x-10y+52=0$

| TYPE III |

TO FIND CENTRE AND RADIUS WHEN GENERAL EQUATION OF CIRCLE IS GIVEN

To find the centre and radius of a circle, whose general equation is given i.e. $x^2+y^2+2gx+2fy+c=0$, *we use the following steps*

Step I Make the coefficients of x^2 and y^2 equal to 1.

Step II Find coordinates of centre (α,β),

where, $\alpha=-\frac{1}{2}$ (coefficient of x) $=-\frac{1}{2}(2g)=-g$

and $\beta=-\frac{1}{2}$ (coefficient of y) $=-\frac{1}{2}(2f)=-f$

Step III Find radius by using the formula,

$$\text{Radius}=\sqrt{\alpha^2+\beta^2-\text{Constant term}}$$

or $\quad \text{Radius}=\sqrt{g^2+f^2-c}$

EXAMPLE |19| Find the centre and radius of the circle given by the equation $2x^2 + 2y^2 + 3x + 4y + \frac{9}{8} = 0.$

Sol. Given equation is $2x^2 + 2y^2 + 3x + 4y + \frac{9}{8} = 0.$

On dividing both sides by 2, we get

$$x^2 + y^2 + \frac{3}{2}x + 2y + \frac{9}{16} = 0 \quad \text{...(i)}$$

From Eq. (i), we have coefficient of $x = \frac{3}{2}$

and coefficient of $y = 2$

$\therefore \quad \alpha = \frac{-1}{2}\left(\frac{3}{2}\right) = -\frac{3}{4}$

and $\quad \beta = \frac{-1}{2}(2) = -1$

Hence, centre $= \left(-\frac{3}{4}, -1\right)$

From Eq. (i), we have constant term $= \frac{9}{16}$

$\therefore$ Radius $= \sqrt{\left(\frac{-3}{4}\right)^2 + (-1)^2 - \frac{9}{16}}$

$= \sqrt{\frac{9}{16} + 1 - \frac{9}{16}} = 1$

EXAMPLE |20| Prove that the radius of the circles $x^2 + y^2 = 1$, $x^2 + y^2 - 2x - 6y = 6$ and $x^2 + y^2 - 4x - 12y = 9$ are in AP.

Sol. Given circles are $x^2 + y^2 - 1 = 0$ ...(i)

$x^2 + y^2 - 2x - 6y - 6 = 0$ ...(ii)

and $x^2 + y^2 - 4x - 12y - 9 = 0$ ...(iii)

Let r_1, r_2 and r_3 be the radii of circles (i), (ii) and (iii), respectively.

We know that the general form of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(iv)}$$

Now, comparing Eq. (i) with Eq. (iv), we get

$2g = 0 \Rightarrow g = 0, 2f = 0$

$\Rightarrow f = 0$ and $c = -1$

$\therefore$ Radius $(r_1) = \sqrt{g^2 + f^2 - c} = \sqrt{0^2 - 0^2 - (-1)} = 1$

On comparing Eq. (ii) with Eq. (iv), we get

$2g = -2 \Rightarrow g = -1, 2f = -6$

$\Rightarrow f = -3$ and $c = -6$

$\therefore$ Radius $(r_2) = \sqrt{g^2 + f^2 - c} = \sqrt{(-1)^2 + (-3)^2 + 6}$

$= \sqrt{1 + 9 + 6} = \sqrt{16} = 4$

Again, comparing Eq. (iii) with Eq. (iv), we get

$2g = -4 \Rightarrow g = -2$

$2f = -12$

$\Rightarrow f = -6$ and $c = -9$

$\therefore$ Radius $(r_3) = \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-6)^2 - (-9)}$

$= \sqrt{4 + 36 + 9} = \sqrt{49} = 7$

Now, $r_2 - r_1 = 4 - 1 = 3$, $r_3 - r_2 = 7 - 4 = 3$

So, r_1, r_2 and r_3 are in arithmetic progression. **Hence proved.**

|TYPE IV|

TO FIND THE EQUATION OF A CIRCLE PASSING THROUGH THREE NON-COLLINEAR POINTS

If we have to find the equation of circle passing through three non-collinear points, then for finding this equation of circle, *we use the following steps*

***Step* I** First, assume the general equation of circle.

i.e. $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

***Step* II** Since, circle passes through three points, so each point will satisfy it.

Put each point in Eq. (i) one-by-one and get three equations.

***Step* III** Now, solve these three equations to get the values of g, f and c and put these values in Eq. (i) to get required equation of circle.

EXAMPLE |21| Find the equation of a circle passing through the points (2, − 6), (6, 4) and (− 3, 1).

Sol. Let equation of the circle passing through the given points be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Since, the circle passes through the point (2, − 6).

On, putting $x = 2, y = -6$ in Eq. (i), we get

$4 + 36 + 4g - 12f + c = 0 \Rightarrow 4g - 12f + c = -40$ ...(ii)

Also, the circle passes through the point (6, 4).

On, putting $x = 6, y = 4$ in Eq. (i), we get

$36 + 16 + 12g + 8f + c = 0$

$\Rightarrow \quad 12g + 8f + c = -52$ ...(iii)

Also the circle passes through the point (− 3, 1).

On putting $x = -3$ and $y = 1$ in Eq. (i), we get

$9 + 1 - 6g + 2f + c = 0 \Rightarrow -6g + 2f + c = -10$ ...(iv)

On subtracting Eq. (iii) from Eq. (ii), we get

$-8g - 20f = 12$ ...(v)

On subtracting Eq. (iv) from Eq. (iii), we get

$18g + 6f = -42$ ...(vi)

On solving Eqs. (v) and (vi) for g and f, we get

$$g = -\frac{32}{13}, f = \frac{5}{13}$$

On putting the values of g and f in Eq. (ii), we get

$$c = -\frac{332}{13}$$

Now, putting $g = -\frac{32}{13}, f = \frac{5}{13}$

and $c = -\frac{332}{13}$ in Eq. (i), we get

$$x^2 + y^2 - \frac{64}{13}x + \frac{10}{13}y - \frac{332}{13} = 0$$

$\Rightarrow \quad 13x^2 + 13y^2 - 64x + 10y - 332 = 0$

which is the required equation of circle.

EXAMPLE |22| Show that the points (5, 5), (6, 4), (– 2, 4) and (7, 1) are concyclic, i.e. all lie on the same circle. Find the equation, centre and radius of this circle.

Sol. Let the equation of the circle passing through the points (5, 5), (6, 4) and (7, 1) be given by

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Then, each of these points must satisfies Eq. (i).

$\therefore \quad 25 + 25 + 10g + 10f + c = 0$

$\Rightarrow \quad 10g + 10f + c = -50 \quad \text{...(ii)}$

$36 + 16 + 12g + 8f + c = 0$

$\Rightarrow \quad 12g + 8f + c = -52 \quad \text{...(iii)}$

and $\quad 49 + 1 + 14g + 2f + c = 0$

$\Rightarrow \quad 14g + 2f + c = -50 \quad \text{...(iv)}$

On solving Eqs. (ii), (iii) and (iv) simultaneously, we get

$g = -2, f = -1$ and $c = -20$

On putting the values of g, f and c in Eq. (i), we get

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

On putting $x = -2$ and $y = 4$ in the above circle, we get

$$4 + 16 + 8 - 8 - 20 = 0$$

This shows that the point (– 2, 4) also lies on the circle.

Hence, the points (5, 5), (6, 4), (– 2, 4) and (7, 1) are concyclic and equation of this circle is

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

Its centre is $(-g, -f) = (2, 1)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 1 + 20} = 5$ units

|TYPE V|

DIFFERENT PROBLEMS BASED ON CIRCLE

EXAMPLE |23| If the line $y = \sqrt{3}x + k$ touches the circle $x^2 + y^2 = 16$, then find the value of k.

Sol. Given, $y = \sqrt{3}x + k$ touches the circle $x^2 + y^2 = 16$.

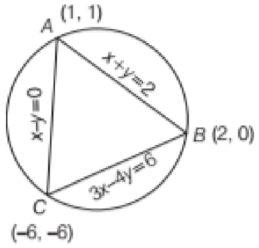
So, length of perpendicular distance from the centre of the given circle i.e. (0, 0) on $y = \sqrt{3}x + k$ is equal to radius of circle.

$$\therefore \quad 4 = \left|\frac{0 + 0 + k}{\sqrt{1 + 3}}\right|$$

$$\Rightarrow \quad 4 = \left|\frac{k}{2}\right| \Rightarrow \frac{k}{2} = \pm 4 \Rightarrow k = \pm 8$$

EXAMPLE |24| Find the equation of the circle passing through the vertices of a triangle whose sides are represented by the equations $x + y = 2$, $3x - 4y = 6$ and $x - y = 0$.

Sol. Let the sides AB, BC and CA of ΔABC be represented by the equations $x + y = 2$, $3x - 4y = 6$ and $x - y = 0$, respectively.



On solving $x + y = 2$ and $3x - 4y = 6$, we get $B(2, 0)$

On solving $3x - 4y = 6$ and $x - y = 0$, we get $C(-6, -6)$

On solving $x + y = 2$ and $x - y = 0$, we get $A(1, 1)$

Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Since, it passes through $A(1, 1)$, $B(2, 0)$ and $C(-6, -6)$, so each of these points satisfy Eq. (i).

$\therefore \quad 1^2 + 1^2 + 2g + 2f + c = 0$

$\Rightarrow \quad 2g + 2f + c = -2 \quad \text{...(ii)}$

$2^2 + 0^2 + 4g + c = 0$

$\Rightarrow \quad 4g + c + 4 = 0 \quad \text{...(iii)}$

and $(-6)^2 + (-6)^2 - 12g - 12f + c = 0$

$\Rightarrow \quad 12g + 12f - c = 72 \quad \text{...(iv)}$

On subtracting Eqs. (ii) from (iii), we get

$2g - 2f + 2 = 0 \Rightarrow g - f = -1 \quad \text{...(v)}$

On adding Eq. (iv) and Eq. (iii), we get

$16g + 12f - 68 = 0 \Rightarrow 4g + 3f = 17 \quad \text{...(vi)}$

On solving Eqs. (v) and (vi), we get $g = 2$ and $f = 3$

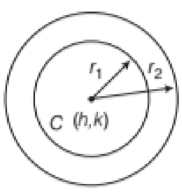
On putting $g = 2$ in Eq. (iii), we get $c = -12$

Hence, the required equation of circle is

$$x^2 + y^2 + 4x + 6y - 12 = 0.$$

Concentric Circles

Two circles having the same centre $C(h, k)$ but different radii r_1 and r_2, are called concentric circles.



Thus, the circles $(x - h)^2 + (y - k)^2 = r_1^2$ and $(x - h)^2 + (y - k)^2 = r_2^2, r_1 \neq r_2$ are concentric circles.

EXAMPLE |25| Find the equation of the circle which passes through the centre of circle $x^2 + y^2 + 8x + 10y - 7 = 0$ and is concentric with the circle $2x^2 + 2y^2 - 8x - 12y - 9 = 0$.

Sol. Given, circles are $x^2 + y^2 + 8x + 10y - 7 = 0$ ...(i)

and $2x^2 + 2y^2 - 8x - 12y - 9 = 0$ ...(ii)

Centre of circle (i) is $C_1(-4, -5)$.

Equation of any circle concentric with circle (ii) is

$$2x^2 + 2y^2 - 8x - 12y + c = 0 \quad \text{...(iii)}$$

If this circle passes through $(-4, -5)$, then

$$2(-4)^2 + 2(-5)^2 - 8(-4) - 12(-5) + c = 0$$

$$\Rightarrow \quad c = -174$$

On substituting the value of c in Eq. (iii), we get

$$2x^2 + 2y^2 - 8x - 12y - 174 = 0$$

Hence, the equation of required circle is

$$2x^2 + 2y^2 - 8x - 12y - 174 = 0$$

or $$x^2 + y^2 - 4x - 6y - 87 = 0$$

Diameter Form of the Equation of a Circle

Let (x_1, y_1) and (x_2, y_2) be the end points of the diameter of a circle. Then, equation of circle drawn on the diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

EXAMPLE |26| Find the equation of the circle, whose end points of a diameter are $A(1, 5)$ and $B(-1, 3)$.

Sol. Given, end points of a diameter are $A(1, 5)$ and $B(-1, 3)$.

We know that, the equation of circle, the end points of one of whose diameter are (x_1, y_1) and (x_2, y_2) is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Hence, the required equation of circle is

$$(x - 1)(x + 1) + (y - 5)(y - 3) = 0$$

$$\Rightarrow \quad x^2 - 1 + y^2 - 8y + 15 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 8y + 14 = 0$$

EXAMPLE |27| Find the equation of the circle, if the end points of whose diameter are the centres of the circles $x^2 + y^2 + 6x - 14y - 1 = 0$ and $x^2 + y^2 - 4x + 10y - 2 = 0$.

Sol. Given circles are

$$x^2 + y^2 + 6x - 14y - 1 = 0$$

and $$x^2 + y^2 - 4x + 10y - 2 = 0$$

On comparing the above equation with $x^2 + y^2 + 2gx + 2fy + c = 0$ one-by-one, we get the centres of given circles $(-3, 7)$ and $(2, -5)$, respectively.

Since, the points $(-3, 7)$ and $(2, -5)$ are end points of the diameter of the required circle.

Hence, the equation of circle is

$$(x + 3)(x - 2) + (y - 7)(y + 5) = 0$$

$$\Rightarrow \quad x^2 + y^2 + x - 2y - 41 = 0$$

EXAMPLE |28| The abscissa of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter.

Sol. Given equations are

$$x^2 + 2ax - b^2 = 0 \text{ and } x^2 + 2px - q^2 = 0.$$

Let the roots of $x^2 + 2ax - b^2 = 0$ be x_1 and x_2.

Then, $x_1 + x_2 = -2a$ and $x_1x_2 = -b^2$

and the roots of $x^2 + 2px - q^2 = 0$ are y_1 and y_2.

Then, $y_1 + y_2 = -2p$ and $y_1y_2 = -q^2$

Let $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$

Now, equation of circle whose diameter is AB, will be

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow x^2 - (x_1 + x_2)x + x_1x_2 + y^2 - (y_1 + y_2)y + y_1y_2 = 0$$

On substituting the values of $x_1 + x_2$, x_1x_2, $y_1 + y_2$ and y_1y_2, we get

$$x^2 + 2ax - b^2 + y^2 + 2py - q^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$$

which is the required equation of circle.

and its radius $= \sqrt{(a)^2 + (p)^2 + b^2 + q^2}$

$$= \sqrt{a^2 + p^2 + b^2 + q^2}$$

Position of a Point w.r.t. Circle

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

or $S \equiv (x - h)^2 + (y - k)^2 = r^2$

be the equation of the circle and $P(x_1, y_1)$ be any point in the plane of the circle, then

$$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

or $$S_1 \equiv (x_1 - h)^2 + (y_1 - k)^2 = r^2$$

Case I If $S_1 > 0$, then the point lies *outside* the circle.

Case II If $S_1 = 0$, then the point lies *on* the circle.

Case III If $S_1 < 0$, then the point lies *inside* the circle.

EXAMPLE |29| Does the point $(-2.5, 3.5)$ lie inside, outside or on the circle $x^2 + y^2 = 25$?

Sol. Given equation of circle is

$$S_1 \equiv x^2 + y^2 = 25$$

or $$S_1 \equiv x^2 + y^2 - 25 = 0 \quad \text{...(i)}$$

Now, put $x = -2.5$ and $y = 3.5$ in Eq. (i), we get

$$S_1 = (-2.5)^2 + (3.5)^2 - 25 = 6.25 + 12.25 - 25$$

$$= 18.5 - 25 = -6.5 < 0$$

Since, $S_1 < 0$, so the given point $(-2.5, 3.5)$ lies inside the circle.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The curves circles, ellipses, parabolas and hyperbolas are known as

(a) conic sections (b) curve sections
(c) line sections (d) plane sections

2 Conic sections or more commonly conics are obtained by intersections of a ...A... with a double napped ...B... . Here, A and B respectively are

(a) line, right circular cone
(b) cone, plane
(c) line, cone
(d) plane, right circular cone

3 The equation of the circle is simplest if the centre of the circle is at the

(a) X-axis
(b) origin
(c) Y-axis
(d) None of the above

4 The radius of the circle whose centre is (2, 3) and which passes through the point (5, 7), is

(a) 5 units (b) 4 units
(c) 3 units (d) 1 unit

5 The equation of the circle with centre (– 3, 2) and radius 4, is

(a) $(x-3)^2+(y-2)^2=16$
(b) $(x+3)^2+(y+2)^2=16$
(c) $(x-3)^2+(y+2)^2=16$
(d) $(x+3)^2+(y-2)^2=16$

VERY SHORT ANSWER Type Questions

6 Find the equation of circle with

(i) centre $=(-3,2)$ and radius $=5$
(ii) centre $=(a,a)$ and radius $=a\sqrt{2}$
(iii) centre $=(a,b)$ and radius $=\sqrt{a^2+b^2}$
(iv) centre $=\left(\frac{1}{3},\frac{1}{4}\right)$ and radius $=12$
(v) centre $=(b\sin\alpha, a\cos\alpha)$ and radius $=1$.

7 Find the centre and radius of the circle of the followings.

(i) $(x+5)^2+(y-3)^2=36$
(ii) $x^2+y^2-4x-8y-45=0$
(iii) $x^2+y^2-6x+4y-12=0$.
(iv) $x^2+y^2+8x+10y-8=0$
(v) $x^2+y^2-2x+4y=8$
(vi) $x^2+y^2+6x-10y+16=0$
(vii) $x^2+y^2+8x-10y+16=0$
(viii) $x^2+y^2+10x-8y-36=0$
(ix) $3x^2+3y^2+6x-4y-1=0$

8 Find the equation of the circles whose end points of one of the diameters are

(i) $A(2,-3)$ and $(-3,5)$
(ii) $P(5,-3)$ and $Q(2,-4)$
(iii) $A(p,q)$ and $B(r,s)$.

SHORT ANSWER Type I Questions

9 Find the equation of the circle whose centre is (2, – 5) and which passes through the point (3, 2).

10 Find the equation of the circle whose centre is (2, – 3) and which passes through the intersection of the lines $3x+2y=11$ and $2x+3y=4$.

11 Find the equation of a circle concentric with the circle $2x^2+2y^2+8x+10y-39=0$ and having its area equal to 16π sq units.

12 If one end of a diameter of circle $x^2+y^2-4x-6y+11=0$ is (3, 4), then find the coordinate of the other end of the diameter.

13 Find the equation of the circle which touches the both axes in the first quadrant and whose radius is a.

14 Find the equation of the circle having centre (3, – 4) and touching the line $5x+12y-19=0$.

15 Find the area of the circle having centre at (1, 2) and passing through (4, 6).

16 A circle of radius 5 units touches the coordinate axes in the first quadrant. If the circle makes one complete roll on X-axis along the positive direction of X-axis, then find its equation in the new position.

17 If $lx+my=1$ touches the circle $x^2+y^2=a^2$, then prove that the point (l,m) lies on the circle $x^2+y^2=a^{-2}$.

SHORT ANSWER Type II Questions

18 Find the equation of the circle which passes through the points $(2,-2)$ and $(3,4)$ and whose centre lies on the line $x + y = 2$.

19 Find the equation of the circle passes through the points (2, 3) and (4, 5) and the centre lies on the straight line $y - 4x + 3 = 0$.

20 Find the equation of circle which is circumscribed about the triangle whose vertices are $(-2, 3)$, $(5, 2)$ and $(6, -1)$.

21 Find the equation of circle circumscribing the triangle whose sides are the lines $y = x + 2$, $4x - 3y = 0$ and $3x - 2y = 0$.

22 Find the equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median of length is $3a$.

23 Show that the points (9, 1), (7, 9), $(-2, 12)$ and (6, 10) are concyclic.

24 If $y = 2x$ is a chord of the circle $x^2 + y^2 - 10x = 0$, then find the equation of a circle with this chord as diameter.

25 Find the equation of a circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has double its area.

26 Show that the points (x, y) given by $x = \frac{2at}{1+t^2}$ and $y = \frac{a(1-t^2)}{1+t^2}$ lies on a circle for all real values of t such that $-1 \le t \le 1$, where a is any given real numbers.

27 Find the equation of the circle concentric with the circle $x^2 + y^2 + 4x + 6y + 11 = 0$ and passing through the point $P(5, 4)$.

28 Does the point $(-1.5, 2.5)$ lies inside or outside or on the circle $x^2 + y^2 = 25$?

HINTS & ANSWERS

1. (a) The curves circles, ellipses, parabolas and hyperbolas are known as conic sections.

2. (d) Conic sections or more commonly conics are obtained by intersections of a plane with a double napped right circular cone.

3. (b) The equation of the circle is simplest if the centre circle is at the origin.

4. (a) The radius of circle

$$= \text{Distance between } (2, 3) \text{ and } (5, 7)$$
$$= \sqrt{(5-2)^2 + (7-3)^2}$$
$$= \sqrt{3^2 + 4^2} = 5 \text{ units}$$

5. (d) Here, $h = -3$, $k = 2$ and $r = 4$.

Therefore, the equation of the required circle is
$$(x + 3)^2 + (y - 2)^2 = 16$$

6. (i) Let the centre of circle be $(h, k) \equiv (-3, 2)$ and radius, $r = 5$.

On putting the values of h, k and r in the standard equation of circle i.e. $(x - h)^2 + (y - k)^2 = r^2$, we get
$$(x + 3)^2 + (y - 2)^2 = (5)^2$$

Ans. $x^2 + y^2 + 6x - 4y - 12 = 0$

(ii) Solve as part (i). **Ans.** $x^2 + y^2 - 2ax - 2ay = 0$

(iii) Solve as part (i). **Ans.** $x^2 + y^2 - 2ax - 2by = 0$

(iv) Solve as part (i).

Ans. $144x^2 + 144y^2 - 96x - 72y - 2071 = 0$

(v) Solve as part (i).

Ans. $x^2 + y^2 - 2bx \sin\alpha - 2ay \cos\alpha + a^2 \cos^2\alpha + b^2 \sin^2\alpha - 1 = 0$

7. (i) Given, $(x + 5)^2 + (y - 3)^2 = 36$

On comparing the above equation with $(x - h)^2 + (y - k)^2 = r^2$, we get
$$h = -5,\ k = 3 \text{ and } r = 6$$
$\therefore$ Centre $(h, k) \equiv (-5, 3)$ and radius $(r) = 6$

(ii) Given, $x^2 + y^2 - 4x - 8y - 45 = 0$

$\Rightarrow (x^2 - 4x) + (y^2 - 8y) = 45$

$\Rightarrow (x^2 - 4x + 4) + (y^2 - 8y + 16) = 4 + 16 + 45$

$\Rightarrow (x - 2)^2 + (y - 4)^2 = (\sqrt{65})^2$

On compare the above equation with $(x - h)^2 + (y - k)^2 = r^2$, we get
$$h = 2,\ k = 4 \text{ and } r = \sqrt{65}$$
$\therefore$ Centre $(h, k) \equiv (2, 4)$ and radius $= \sqrt{65}$

(iii) Solve as part (ii).

Ans. Centre $\equiv (3, -2)$, radius $= 5$

(iv) Solve as part (ii). **Ans.** $(-4, -5)$, 7

(v) Solve as part (ii). **Ans.** $(1, -2)$, $\sqrt{13}$.

(vi) Solve as part (ii). **Ans.** $(-3, 4)$, $\sqrt{33}$

(vii) Solve as part (ii). **Ans.** $(-4, 5)$, 5

(viii) Solve as part (ii). **Ans.** $(-5, 4)$, $\sqrt{77}$

(ix) Solve as part (ii). **Ans.** Centre $\equiv \left(-1, \frac{2}{3}\right)$, radius $= \frac{4}{3}$

8. (i) Required equation of circle is $(x - 2)(x + 3) + (y + 3)(y - 5) = 0$.

Ans. $x^2 + y^2 + x - 2y - 21 = 0$

(ii) Solve as part (i). **Ans.** $x^2 + y^2 - 7x + 7y + 22 = 0$

(iii) Solve as part (i).

Ans. $(x - p)(x - r) + (y - q)(y - s) = 0$

9. Let the equation of circle be $(x-2)^2+(y+5)^2=r^2$

$\Rightarrow \quad x^2+4-4x+y^2+25+10y=r^2$

$\Rightarrow \quad x^2+y^2-4x+10y+29-r^2=0$

Since, it passes through (3, 2).

$\therefore \quad 9+4-12+20+29-r^2=0 \Rightarrow r=5\sqrt{2}$

Ans. $x^2+y^2-4x+10y-50=0$

10. Let the equation of circle be $(x-2)^2+(y+3)^2=r^2$

Intersection point is (5, − 2).

Ans. $x^2+y^2-4x+6y-10=0$

11. Centre of given circle $x^2+y^2+4x+5y-\frac{39}{2}=0$ is $\left(-2,-\frac{5}{2}\right)$ and area of circle, $\pi r^2=16\pi$

$\Rightarrow \quad r^2=16$

Then, the equation of required circle is

$$(x+2)^2+\left(y+\frac{5}{2}\right)^2=16$$

Ans. $4x^2+4y^2+16x+20y-23=0$

12. Centre of circle is mid-point of end point of diameter.

Here, centre = (2, 3)

Let the other end be (x, y).

Then, $\frac{x+3}{2}=2, \frac{y+4}{2}=3$

Ans. (1, 2)

13. Required equation of circle is $(x-a)^2+(y-a)^2=a^2$

Ans. $x^2+y^2-2ax-2ay+a^2=0$

14. Centre of circle = (3, −4)

Radius of circle = Perpendicular distance of line $5x+12y-19=0$ from (3, −4).

$$=\left|\frac{5(3)+12(-4)-19}{\sqrt{25+144}}\right|=\left|\frac{-52}{13}\right|=4$$

$\therefore$ Required equation of circle is

$$(x-3)^2+(y+4)^2=(4)^2.$$

Ans. $x^2+y^2-6x+8y+9=0$

15. Let equation of circle having centre (1, 2) be

$$(x-1)^2+(y-2)^2=r^2$$

Since, it passes through (4, 6).

$\therefore \quad (4-1)^2+(6-2)^2=r^2 \Rightarrow r^2=9+16=25 \Rightarrow r=5$

Ans. 25π sq units

16. Let C be the centre of the circle in its initial position and D be its centre in the new position.

Then, $C\equiv(5,5)$ and $D=(5+10\pi,5)$

Now, centre of the circle in the new position is $(5+10\pi, 5)$ and its radius is 5, therefore its equation will be

$$(x-5-10\pi)^2+(y-5)^2=5^2$$

$\Rightarrow \quad x^2+25+100\pi^2-10x-20\pi x+100\pi+y^2+25-10y=25$

Ans. $x^2+y^2-10(2\pi+1)x-10y+100\pi^2+100\pi+25=0$

17. Given, $lx+my=1$ touches the circle $x^2+y^2=a^2$.

Then, length of perpendicular distance from the centre of the given circle i.e. (0, 0) on $lx+my+1=0$ is equal to radius.

Then, $a=\left|\frac{0-0+1}{\sqrt{l^2+m^2}}\right| \Rightarrow a^2=\frac{1}{l^2+m^2} \Rightarrow l^2+m^2=a^{-2}$

18. Let the equation of circle with centre (h, k) and radius r be

$$(x-h)^2+(y-k)^2=r^2 \quad \text{...(i)}$$

Since, circle passes through the points (2,−2) and (3,4), so the points (2, − 2) and (3, 4) will lie on Eq. (i).

$\therefore \quad (2-h)^2+(-2-k)^2=r^2 \quad$ ...(ii)

and $\quad (3-h)^2+(4-k)^2=r^2 \quad$ ...(iii)

Now, from Eqs. (ii) and (iii), we get

$$(2-h)^2+(-2-k)^2=(3-h)^2+(4-k)^2$$

$\Rightarrow \quad 2h+12k=17 \quad$ ...(iv)

Also, given that centre (h, k) lies on $x+y=2$. So, it will satisfy it.

$\therefore \quad h+k=2 \quad$ ...(v)

On solving Eqs. (iv) and (v), we get

$$h=0.7, k=1.3$$

Now, from Eq. (ii), we get

$$r^2=(2-0.7)^2+(-2-1.3)^2=1.69+10.89=12.58$$

Now, put the values of h, k and r^2 in Eq. (i), we get the answer.

Ans. $(x-0.7)^2+(y-1.3)^2=12.58$

19. Solve as Q. 18. **Ans.** $x^2+y^2-4x-10y+25=0$

20. Circle circumscribed about the triangle means the circle passes through the vertices. Let (h, k) be the centre and r be the radius of circle.

Then, $\quad (-2-h)^2+(3-k)^2=r^2 \quad$ ...(i)

$(5-h)^2+(2-k)^2=r^2 \quad$ ...(ii)

and $\quad (6-h)^2+(-1-k)^2=r^2 \quad$ ...(iii)

On solving Eqs. (i), (ii) and (iii), we get $h=1, k=-1$ and $r=5$ **Ans.** $x^2+y^2-2x+2y-23=0$

21. Solve as Example 24. **Ans.** $x^2+y^2-46x+22y=0$

22. Centroid of the triangle coincides with the centre of circle and the radius of circle is 2/3 of the length of median.

Ans. $x^2+y^2=4a^2$

24. The points of intersection of the given chord and the given circle are obtained by simultaneously solving $y=2x$ and $x^2+y^2-10x=0$.

On putting $y=2x$ in $x^2+y^2-10x=0$, we get

$$x^2+4x^2-10x=0$$

$\Rightarrow \quad 5x(x-2)=0$

$\Rightarrow \quad x=0$ or $x=2$

Now, $\quad x=0 \Rightarrow y=0$ and $x=2, y=4$

Hence, the required equation of circle is

$$(x-0)(x-2)+(y-0)(y-4)=0$$

$$x^2+y^2-2x-4y=0$$

25. Given equation of the circle is

$$x^2 + y^2 - 6x + 12y + 15 = 0$$

Centre of this circle = $(3, -6)$

Radius of this circle = $\sqrt{9 + 36 - 15} = \sqrt{30}$

The required circle is concentric to the given circle.

$\therefore$ Its centre is also coincide. So, its centre is $(3, -6)$.

Let r be the radius of the required circle.

Then, according to the question,

$$\pi r^2 = 2 \times \pi(\sqrt{30})^2 \Rightarrow r^2 = 60$$

Hence, the equation of the required circle is

$$(x-3)^2 + (y+6)^2 = 60$$

or $\quad x^2 + y^2 - 6x + 12y - 15 = 0$.

26. Given, $x = \dfrac{2at}{1+t^2}$ and $y = \dfrac{a(1-t^2)}{1+t^2}$

On squaring and adding, we get

$$x^2 + y^2 = \left(\frac{2at}{1+t^2}\right)^2 + \left(\frac{a(1-t^2)}{1+t^2}\right)^2$$

$$= \frac{4a^2t^2}{(1+t^2)^2} + \frac{a^2(1-t^2)^2}{(1+t^2)^2} = \frac{a^2(4t^2+1+t^4-2t^2)}{(1+t^2)^2}$$

$$= \frac{a^2[t^4+1+2t^2]}{(1+t^2)^2} = \frac{a^2(t^2+1)^2}{(t^2+1)^2} = a^2$$

$$\Rightarrow x^2 + y^2 = a^2$$

27. Centre of the required circle is $C(-2, -3)$.

Circle passes through the point $P(5, 4)$.

$\therefore \quad$ Radius $= |CP| = \sqrt{(5+2)^2 + (4+3)^2} = 7\sqrt{2}$

Ans. $(x+2)^2 + (y+3)^2 = 98$

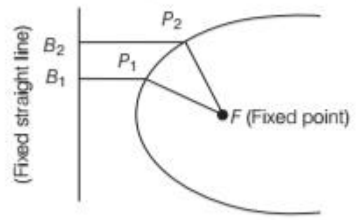
28. Solve as Example 29. **Ans.** Inside the circle.

|TOPIC 2|
Parabola

The word 'Parabola' is derived from two words '**Para**' and '**Bola**'. Para means 'for' and bola means 'throwing' i.e. the shape described when we throw a ball in the air.

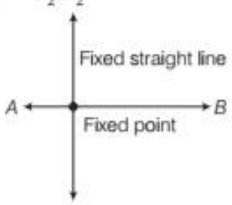
Definition

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point is always equal to its distance from a fixed straight line in the same plane.



From the figure, $P_1F = P_1B_1$ and $P_2F = P_2B_2$

Here, the fixed line is called the **directrix** and the fixed point is called **focus** of the parabola. A line through the focus and perpendicular to the directrix is called the **axis** of the parabola and point of intersection of parabola with the axis is called the **vertex** of the parabola.



If the fixed point lies on the fixed line, then the set of points in the plane, which are equidistant from the fixed point and the fixed line is the straight line through the fixed point and perpendicular to the straight line. The straight line AB is called a degenerate case of the parabola.

Note In case of parabola, eccentricity (e) is 1.

Standard Equations of Parabola

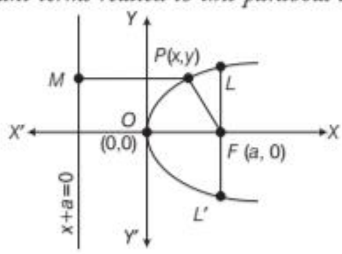
If the equation of the parabola having vertex at origin and the axis of symmetry is either X-axis or Y-axis, then the equations of parabola is said to be in simplest form.

There are four standard equations of parabola which are given below.

RIGHT HANDED PARABOLA

If parabola opens towards right side, then it is called right handed parabola. The equation of right handed parabola is of the form $y^2 = 4ax$, $a > 0$

Some important terms related to this parabola are



(i) Vertex is $O(0, 0)$.

(ii) Axis is the line $y = 0$.

(iii) Focus is $F(a, 0)$.

(iv) Directrix is the line $x + a = 0$.

(v) Latusrectum of a parabola is a line segment through focus and perpendicular to the axis of the parabola, whose end points lie on the parabola i.e. $LL' = 4a$.

(vi) Coordinates of latusrectum $= (a, \pm 2a)$

(vii) Equation of latusrectum is $x = a$ or $x - a = 0$.

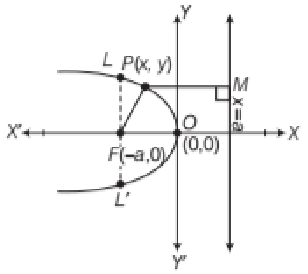
(viii) It is symmetrical about X-axis.

LEFT HANDED PARABOLA

If parabola opens towards left side, then it is called left handed parabola. The equation of the left handed parabola is of the form

$$\boxed{y^2 = -4ax,\ a > 0}$$

Some important terms related to this parabola are



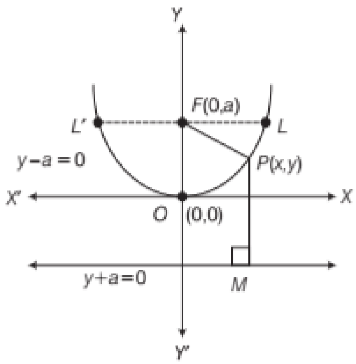
(i) Vertex is $O\,(0, 0)$.
(ii) Axis is the line $y = 0$.
(iii) Focus is $F(-a, 0)$.
(iv) Directrix is the line $x - a = 0$.
(v) Length of the latusrectum is $4a$.
(vi) Coordinates of latusrectum $= (-a, \pm 2a)$
(vii) Equation of latusrectum is $x = -a$ or $x + a = 0$.
(viii) It is symmetrical about X-axis.

UPWARD PARABOLA

If parabola opens upward, then it is called upward parabola. The equation of the upward parabola is of the form

$$\boxed{x^2 = 4ay,\ a > 0}$$

Some important terms related to this parabola are



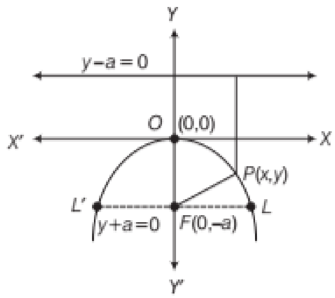
(i) Focus is $F(0, a)$.
(ii) Vertex is $O(0, 0)$.
(iii) Directrix is the line $y + a = 0$.
(iv) Axis is the line $x = 0$.
(v) Length of latusrectum is $4a$.
(vi) Coordinates of latusrectum $= (\pm 2a, a)$
(vii) Equation of latusrectum is $y = a$ or $y - a = 0$.
(viii) It is symmetrical about Y-axis.

DOWNWARD PARABOLA

If parabola opens downwards, it is called downward parabola. The equation of downward parabola is of the form

$$\boxed{x^2 = -4ay,\ a > 0}$$

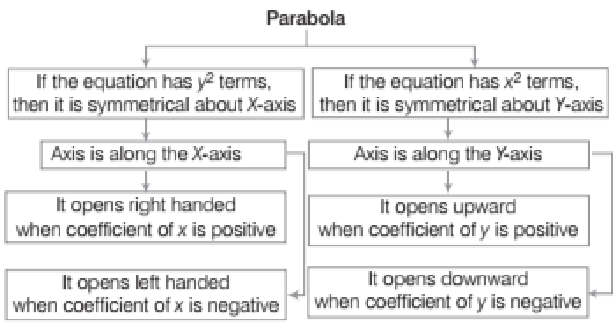
Some important terms related to this parabola are



(i) Focus is $F(0, -a)$.
(ii) Vertex is $O(0, 0)$.
(iii) Directrix is the line $y - a = 0$.
(iv) Axis is the line $x = 0$.
(v) Length of latusrectum is $4a$.
(vi) Coordinates of latusrectum $= (\pm 2a, (-a))$
(vii) Equation of latusrectum is $y = -a$ or $y + a = 0$.
(viii) It is symmetrical about Y-axis.

OBSERVATIONS FROM STANDARD EQUATIONS

From the standard equations of the parabola, following observations can be explained through flow chart.



Note

The standard equations of parabolas have focus on one of the coordinate axis, vertex at the origin and the directrix is parallel to the other coordinate axis.

Table for Different Types of Parabola

Parabola	Vertex	Focus	Latusrectum	Coordinates of latusrectum	Axis	Directrix	Symmetry
$y^2 = 4ax$	(0, 0)	(a, 0)	4a	$(a, \pm 2a)$	$y = 0$	$x = -a$	X-axis
$y^2 = -4ax$	(0, 0)	(−a, 0)	4a	$(-a, \pm 2a)$	$y = 0$	$x = a$	X-axis
$x^2 = 4ay$	(0, 0)	(0, a)	4a	$(\pm 2a, a)$	$x = 0$	$y = -a$	Y-axis
$x^2 = -4ay$	(0, 0)	(0, −a)	4a	$(\pm 2a, -a)$	$x = 0$	$y = a$	Y-axis

Different Types of Questions Based on Parabola

|TYPE I|

TO FIND DIFFERENT TERMS RELATED TO PARALBOLA WHEN STANDARD EQUATION IS GIVEN

In this type of questions, equation of parabola is given and we have to find different terms related to it as coordinates of focus, axis, directrix, length of latusrectum, etc.

EXAMPLE |1| In each of the following questions, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latusrectum.

(i) $y^2 = 12x$ (ii) $y^2 = -8x$
(iii) $y^2 = 10x$ (iv) $x^2 = -9y$

Sol. (i) Given, equation of parabola is $y^2 = 12x$ which is of the form, $y^2 = 4ax$ i.e focus lies on the positive direction of X-axis.

Here, $4a = 12 \Rightarrow a = 3$

$\therefore$ Focus $= (a, 0) = (3, 0)$

Axis = X-axis

Directrix, $x = -a \Rightarrow x = -3$

and length of latusrectum $= 4a = 4 \times 3 = 12$

(ii) Given, equation of parabola is $y^2 = -8x$ which is of the form $y^2 = -4ax$ i.e. focus lies on the negative direction of X-axis.

Here, $4a = 8 \Rightarrow a = 2$

$\therefore$ Focus $= (-a, 0) = (-2, 0)$

Axis = X-axis

Directrix, $x = a \Rightarrow x = 2$

and length of latusrectum $= 4a = 4 \times 2 = 8$

(iii) Given, equation of parabola is $y^2 = 10x$, which is of the form $y^2 = 4ax$ i.e. focus lies on the positive direction of X-axis.

Here, $4a = 10 \Rightarrow a = \frac{5}{2}$

$\therefore$ Focus $= (a, 0) = \left(\frac{5}{2}, 0\right)$

Axis = X-axis

Directrix, $x = -a \Rightarrow x = -\frac{5}{2}$

and length of latusrectum $= 4a = 4 \times \frac{5}{2} = 10$

(iv) Given, equation of parabola is $x^2 = -9y$, which is of the form $x^2 = -4ay$ i.e. focus lies on the negative direction of Y-axis.

Here, $4a = 9$

$\Rightarrow a = \frac{9}{4}$

$\therefore$ Focus $= (0, -a) = \left(0, -\frac{9}{4}\right)$

Axis = Y-axis

Directrix, $y = a \Rightarrow y = \frac{9}{4}$

and length of latusrectum $= 4a = 9$

|TYPE II|

TO FIND THE EQUATION OF PARABOLA IN DIFFERENT CASES

***Case* I When Focus and Directrix are given**

If focus F and directrix of a parabola are given, *the we can find the equation of parabola with the help of following steps*

***Step* I** Take any point $P(x, y)$ on the parabola.

***Step* II** Find the distance between the point P and the focus F by distance formula.

***Step* III** Also, find the distance of the point from the directrix by distance formula.

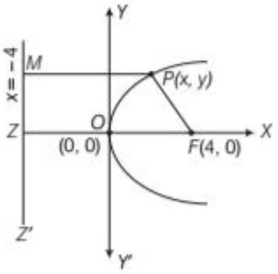
***Step* IV** Equate the distances calculated in step II and step III (by definition of parabola).

The expression so obtained is the required equation of parabola.

EXAMPLE |2| Find the equation of the parabola, whose focus is the point (4, 0) and whose directrix is $x = -4$. Also, find the length of latusrectum.

Sol. Here, focus is $(4, 0)$ and directrix is $x + 4 = 0$.

Let $P(x, y)$ be any moving point, then draw $PM \perp ZZ'$ from P to the directrix.



Now, $FP = PM$ [by definition of parabola]

$\Rightarrow \quad FP^2 = PM^2$

$$\Rightarrow \quad (x-4)^2 + (y-0)^2 = \frac{(x+4)^2}{(\sqrt{1})^2}$$

$\Rightarrow \quad x^2 - 8x + 16 + y^2 = x^2 + 8x + 16$

$\Rightarrow \quad y^2 = 16x$

which is the required equation of parabola.

$\therefore$ Length of latusrectum $= 16$

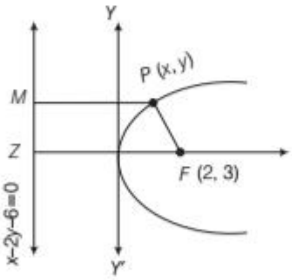
EXAMPLE |3| Find the equation of parabola whose focus is $(2, 3)$ and directrix is $x - 2y - 6 = 0$.

Sol. Let $P(x, y)$ be any point on the parabola.

Distance of $P(x, y)$ from the focus $(2, 3)$
$=$ Distance of $P(x, y)$ from the directrix $x - 2y - 6 = 0$
[by definition of parabola]

$PF = PM$

$\Rightarrow \quad PF^2 = PM^2$



$$\Rightarrow \quad (x-2)^2 + (y-3)^2 = \left|\frac{x - 2y - 6}{\sqrt{1^2 + 2^2}}\right|^2$$

$$\Rightarrow \quad (x-2)^2 + (y-3)^2 = \frac{(x - 2y - 6)^2}{5}$$

$\Rightarrow \quad 5[(x-2)^2 + (y-3)^2] = (x - 2y - 6)^2$

$\Rightarrow \quad 5(x^2 - 4x + 4 + y^2 - 6y + 9)$
$= x^2 + 4y^2 + 36 - 12x - 4xy + 24y$

$\Rightarrow \quad 4x^2 + y^2 + 4xy - 8x - 54y + 29 = 0$

which is the required equation of parabola.

***Case* II When Vertex and Focus are given**

If vertex V and focus F of a parabola are given, *then we use the following steps for finding the equation of parabola.*

***Step* I** Find the point of intersection of axis and directrix by using the mid-point formula.

***Step* II** Find the slope of the line formed by joining focus and vertex, i.e. axis by the formula $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$.

***Step* III** Find the slope of the directrix by the formula $m_1 \cdot m_2 = -1$, as axis is perpendicular to directrix, where m_1 is the slope of axis of the parabola and m_2 is the slope of directrix.

***Step* IV** Write down the equation of directrix by using slope point formula.

***Step* V** Now, focus and directrix are known, so find the equation of parabola by using the definition of parabola, $FP = PN$.

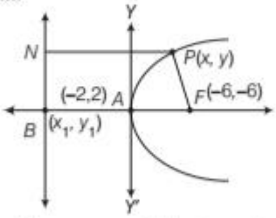
EXAMPLE |4| Find out the equation of parabola, if the focus is at $(-6, -6)$ and the vertex is at $(-2, 2)$.

Sol. Given, focus is at $F(-6, -6)$ and vertex is at $A(-2, 2)$.

Let $B(x_1, y_1)$ be the coordinates of the point of intersection of axis and directrix. Then, $A(-2, 2)$ is the mid-point of line segment joining $F(-6, -6)$ and $B(x_1, y_1)$.

$$\therefore \quad -2 = \frac{x_1 - 6}{2} \Rightarrow x_1 = 2 \text{ and } 2 = \frac{y_1 - 6}{2} \Rightarrow y_1 = 10$$

So, the point $B(2, 10)$ is the point of intersection of axis and directrix.



Now, slope of line segment joining vertex and focus is

$$m_1 = \frac{-6-2}{-6+2} = \frac{-8}{-4} = 2$$

$$\therefore \text{ Slope of directrix, } m_2 = \frac{-1}{2} \quad \left[\begin{array}{l} \because AF \perp NB, \text{ so } m_1 m_2 = -1 \\ \Rightarrow m_2 = \dfrac{-1}{m_1} \end{array}\right]$$

Then, the equation of directrix is $y - 10 = \dfrac{-1}{2}(x - 2)$

$\Rightarrow \quad 2y - 20 = -x + 2 \Rightarrow x + 2y = 22$

Let $P(x, y)$ be any point on parabola and PN be the length of perpendicular from P on directrix and FP be the distance between focus F and point P.

So, $\quad FP = PN \Rightarrow (FP)^2 = (PN)^2$

$$\Rightarrow \quad (x+6)^2 + (y+6)^2 = \left(\frac{x + 2y - 22}{\sqrt{1+4}}\right)^2$$

$\Rightarrow \quad x^2 + 36 + 12x + y^2 + 36 + 12y$

$$= \frac{x^2 + 4y^2 + 484 + 4xy - 44x - 88y}{5}$$

$\Rightarrow \quad 5x^2 + 180 + 60x + 5y^2 + 60y + 180$

$$= x^2 + 4y^2 + 484 + 4xy - 44x - 88y$$

$\Rightarrow \quad 4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$

which is the required equation of parabola.

EXAMPLE |5| Find the equation of parabola when the vertex is at (0, 0) and focus is at (0, − 4).

As vertex and focus lies on Y-axis, so use the equation of parabola in the form of $x^2 = -4ay$.

Sol. Here, the vertex is at (0, 0) and focus is at (0, −4) which lies on Y-axis. So, Y-axis is the axis of the parabola.

∴ Equation of parabola is of the form

$$x^2 = -4ay \Rightarrow x^2 = -4(4)y \qquad [\because a = 4]$$

$$\therefore \quad x^2 = -16y$$

Case III When Vertex and Directrix are Given

If vertex V and equation of directrix of parabola are given, *then we use the following steps for finding the equation of parabola.*

Step I Find the equation of axis of parabola which is perpendicular to the directrix and passing through the vertex.

Step II Find the point of intersection of axis and directrix of the parabola.

Step III Now, vertex is the mid-point of the line joining the focus and intersection point obtained from step II. So, find the focus of parabola by using the mid-point formula.

Step IV Thus, focus and directrix are known. Now, find the equation of parabola by using the definition of parabola i.e. $PF = PM$.

EXAMPLE |6| Find the equation of the parabola whose vertex is at (2, 1) and the directrix is $x - y + 1 = 0$.

Sol. Let $V(2, 1)$ be the vertex and equation of directrix is

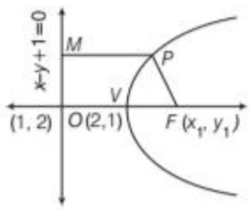
$$x - y + 1 = 0 \qquad \text{...(i)}$$

Then, equation of line perpendicular to directrix i.e. axis is

$$x + y + \lambda = 0$$

Since, this will passes through vertex (2, 1).

$$\therefore \quad 2 + 1 + \lambda = 0 \Rightarrow \lambda = -3$$



So, the equation of line perpendicular to directrix, i.e. equation of axis of parabola is

$$x + y - 3 = 0 \qquad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get $x = 1$, $y = 2$

So, the coordinates of intersection point of directrix and axis is $X(1, 2)$. Let (x_1, y_1) be the coordinates of F, then V is the mid-point of XF.

$$\therefore \quad \frac{x_1 + 1}{2} = 2 \text{ and } \frac{y_1 + 2}{2} = 1$$

$$\Rightarrow \quad x_1 = 3 \text{ and } y_1 = 0$$

So, the coordinates of focus are (3, 0).

Now let $P(x, y)$ be any point on the parabola.

Then by definition,

$$PF = PM \Rightarrow PF^2 = PM^2$$

$$\Rightarrow \quad (x-3)^2 + y^2 = \frac{(x - y + 1)^2}{2}$$

$\Rightarrow 2(x^2 - 6x + 9 + y^2) = x^2 + y^2 + 1 - 2xy + 2x - 2y$

$\Rightarrow \quad x^2 + y^2 + 2xy - 14x + 2y + 17 = 0$

which is the required equation of parabola.

|TYPE III|

TO FIND DIFFERENT TERMS RELATED TO PARABOLA WHEN STANDARD EQUATION IS NOT GIVEN

If the given equation of parabola is not in the standard form, then for finding different terms related to parabola, *we use the following steps*

Step I If term containing xy is absent in the given equation of parabola, then write the equation in one of the following form

(i) $(y - \beta)^2 = 4a(x - \alpha)$, if given equation is quadratic in y and linear in x.

(ii) $(x - \alpha)^2 = 4a(y - \beta)$, if given equation is quadratic in x and linear in y.

Step II Put $Y \equiv y - \beta$ and $X = x - \alpha$ in part (i) and $Y \equiv x - \beta$, $X = y - \beta$ in part (ii), to convert in standard form $Y^2 = 4aX$.

Now, axis of parabola is $Y = 0$.

Coordinates of focus are, $X = a$, $Y = 0$.

Coordinates of vertex are, $X = 0$, $Y = 0$.

Equation of directrix is $X = -a$.

and length of latusrectum $= |4a|$.

Put the values of X and Y and then simplify it to get required values.

EXAMPLE |7| Find the vertex, axis, focus, directrix and length of latusrectum of parabola $y^2 - 8y - x + 19 = 0$.

Sol. Given equation is $y^2 - 8y - x + 19 = 0$

$$\Rightarrow \quad y^2 - 8y + 16 = x - 19 + 16$$

$$\Rightarrow \quad (y-4)^2 = x - 3 \qquad \text{...(i)}$$

Let $\quad y - 4 = Y$ and $x - 3 = X \qquad$ ...(ii)

Then, Eq. (i) becomes $Y^2 = X$ ...(iii)

Now, from Eq. (iii), coordinates of vertex are

$$X = 0 \text{ and } Y = 0$$

$\Rightarrow$ $x - 3 = 0$ and $y - 4 = 0$

[putting the values from Eq. (ii)]

$\Rightarrow$ $x = 3$ and $y = 4$

The equation of axis of parabola (iii) is $Y = 0$.

$\Rightarrow$ $y - 4 = 0 \Rightarrow y = 4$

On comparing Eq. (iii) with $Y^2 = 4aX$, we get

$$4a = 1 \Rightarrow a = \frac{1}{4}$$

Coordinates of focus of parabola (iii) are

$$X = a, Y = 0 \Rightarrow x - 3 = \frac{1}{4},\ y - 4 = 0$$

$$\Rightarrow \quad x = \frac{1}{4} + 3,\ y = 4 \Rightarrow x = \frac{13}{4},\ y = 4$$

Equation of directrix of parabola (iii) is

$$X = -a \Rightarrow x - 3 = -\frac{1}{4}$$

$$\Rightarrow \quad x = -\frac{1}{4} + 3 \Rightarrow x = \frac{11}{4}$$

Length of latusrectum $= |4a| = \left|4 \cdot \frac{1}{4}\right| = 1$

Hence, for given parabola vertex $\equiv (3, 4)$, axis $\equiv y = 4$, focus $\equiv \left(\frac{13}{4}, 4\right)$, directrix $\equiv x = \frac{11}{4}$ and length of latusrectum $= 1$.

|TYPE IV|

MISCELLANEOUS PROBLEMS BASED ON PARABOLA

EXAMPLE |8| Find the coordinates of a point on the parabola $y^2 = 8x$, whose focal distance is 4.

Sol. Given equation of parabola is $y^2 = 8x$.

Here, $a = 2$

$\therefore$ Focus $\equiv (2, 0)$

Now, focal distance $= 4$

We know that focal distance is a distance of any point $P(x, y)$ on the parabola from the focus F.

$\therefore$ PF = Focal distance

$\Rightarrow$ $PF = \sqrt{(x-a)^2 + y^2}$ [by distance formula]

$\Rightarrow$ $4 = \sqrt{(x-2)^2 + 8x}$ $[\because y^2 = 8x]$

$\Rightarrow$ $16 = x^2 - 4x + 4 + 8x$ [on squaring both sides]

$\Rightarrow$ $16 = (x+2)^2 \Rightarrow x + 2 = 4$

[taking positive square root both sides]

$\Rightarrow$ $x = 2$

Now, $y^2 = 8 \times 2 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$

Hence coordinates of a point are (2, 4) and (2, – 4).

Note The distance of a point P on the parabola from the focus is called focal distance of the point P.

EXAMPLE |9| Find the length of the line segment joining the vertex of the parabola $y^2 = 4ax$ and a point on the parabola, where the line segment makes an angle θ to the X-axis.

Sol. Let equation of parabola be

$$y^2 = 4ax \quad ...(i)$$

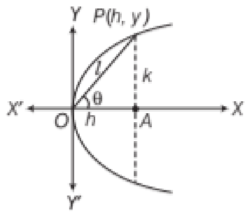
and its vertex is (0, 0).

Again, let any point on the parabola be $P(h, k)$.

On putting the values $x = h$ and $y = k$ in Eq. (i), we get

$$k^2 = 4ah \quad ...(ii)$$

Let $OP(=l)$ be the line segment joining the vertex and point P. Also, it makes an angle θ with the X-axis.



In right angled ΔOAP,

$$\sin\theta = \frac{PA}{OP} \Rightarrow \sin\theta = \frac{k}{l} \Rightarrow k = l\sin\theta$$

and $$\cos\theta = \frac{OA}{OP} \Rightarrow \cos\theta = \frac{h}{l} \Rightarrow h = \cos\theta$$

On substituting the values of h and k in Eq. (ii), we get

$$l^2\sin^2\theta = 4al\cos\theta \Rightarrow l = \frac{4a\cos\theta}{\sin^2\theta}$$

which is the required length.

EXAMPLE |10| Prove that the line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$, if $ln = am^2$.

Sol. Given equation of line is $lx + my + n = 0$.

$$\Rightarrow \quad y = \frac{-lx - n}{m} \quad ...(i)$$

and equation of parabola is $y^2 = 4ax$ ...(ii)

From Eqs. (i) and (ii), we get

$$\left(\frac{-lx-n}{m}\right)^2 = 4ax$$

$\Rightarrow$ $l^2x^2 + 2lxn + n^2 = 4m^2ax$

$\Rightarrow$ $l^2x^2 + 2lxn - 4am^2x + n^2 = 0$

$\Rightarrow$ $l^2x^2 + x(2ln - 4am^2) + n^2 = 0$ ...(iii)

Since, the line $lx + my + n$ touches the parabola.

So, Eq. (iii) have equal roots.

i.e. discriminant $(D) = 0 \Rightarrow B^2 - 4AC = 0$

$\Rightarrow$ $(2ln - 4am^2)^2 - 4l^2n^2 = 0$

$\Rightarrow$ $4l^2n^2 - 16lnam^2 + 16a^2m^4 - 4l^2n^2 = 0$

$\therefore$ $ln = am^2$ **Hence proved.**

EXAMPLE |11| If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$, then find the value of m.

Sol. Given equation of line is $y = mx + 1$ ...(i)

and equation of parabola is $y^2 = 4x$ ...(ii)

On putting the value of y from Eq. (i) in Eq. (ii), we get

$(mx+1)^2 = 4x \Rightarrow m^2x^2 + 1 + 2mx - 4x = 0$

$\Rightarrow \quad x^2(m^2) + x(2m-4) + 1 = 0$

For the tangent, discriminant is zero i.e. $D = 0$.

$\therefore \quad (2m-4)^2 - 4m^2 \times 1 = 0 \quad [\because D = b^2 - 4ac = 0]$

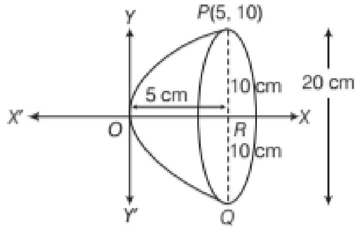
$\Rightarrow \quad m = 1$

| TYPE V |

PROBLEMS BASED ON APPLICATION OF PARABOLA

EXAMPLE |12| If a parabolic reflector is 20 cm in diameter and 5 cm deep. Find the focus.

Sol. Let POQ be the parabolic reflector which is 20 cm in diameter and 5 cm deep.



Then, $PQ = 20$ cm and $OR = 5$ cm, where R is the mid-point of PQ.

We take OX as X-axis and OY as Y-axis. The equation of parabola may be taken as $y^2 = 4ax$. Since, the point $P(5, 10)$ lies on the parabola.

$\therefore \quad 10^2 = 4a(5) \Rightarrow a = 5$

Therefore, the coordinate of the focus are $(a, 0)$ i.e. $(5, 0)$. Hence, the focus is the mid-point of the given diameter.

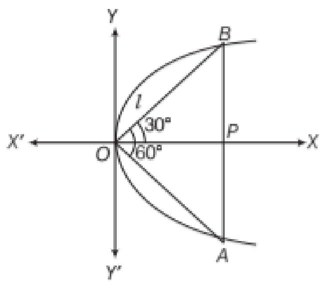
EXAMPLE |13| An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Sol. First, we draw the parabola in the positive side of X-axis and inside that, draw an equilateral ΔOAB.

Let $OB = l = OA = AB \Rightarrow \angle BOA = 60° \Rightarrow \angle BOP = 30°$

In ΔBOP, $\sin 30° = \dfrac{PB}{OB} \Rightarrow \dfrac{1}{2} = \dfrac{PB}{l} \Rightarrow PB = \dfrac{l}{2}$

and $\quad \cos 30° = \dfrac{OP}{OB} \Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{OP}{l} \Rightarrow OP = \dfrac{l\sqrt{3}}{2}$



$\therefore$ Coordinates of $B = (OP, PB) = \left(\dfrac{l\sqrt{3}}{2}, \dfrac{l}{2}\right)$ will satisfy

$$y^2 = 4ax$$

i.e. $\quad \left(\dfrac{l}{2}\right)^2 = \dfrac{4a \times l\sqrt{3}}{2} \Rightarrow \dfrac{l^2}{4} = \dfrac{4al\sqrt{3}}{2} \Rightarrow l = 8\sqrt{3}a$

Hence, the length of side of the triangle is $8\sqrt{3}a$.

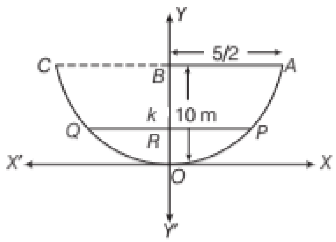
EXAMPLE |14| An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide it is 2 m from the vertex of the parabola?

Sol. Here, axis is vertical, so let arch of parabola is in the form

$$x^2 = 4ay \quad ...(i)$$

Given, $\quad OB = 10$ m

and $\quad AC = 5$ m $\Rightarrow AB = \dfrac{5}{2}$ m $\quad \left[\because AB = BC = \dfrac{BC}{2}\right]$



Hence, coordinates of $A = \left(\dfrac{5}{2}, 10\right)$ will satisfy Eq. (i).

i.e. $\quad \left(\dfrac{5}{2}\right)^2 = 4a \times 10 \Rightarrow \dfrac{25}{4} = 40a \Rightarrow a = \dfrac{5}{32}$

From Eq. (i), $x^2 = 4 \times \dfrac{5}{32} y \Rightarrow x^2 = \dfrac{5}{8} y$

Now, let $\quad OR = 2$ and $PQ = k \Rightarrow RP = \dfrac{k}{2}$

Therefore, $P = \left(\dfrac{k}{2}, 2\right)$ will lie on parabola.

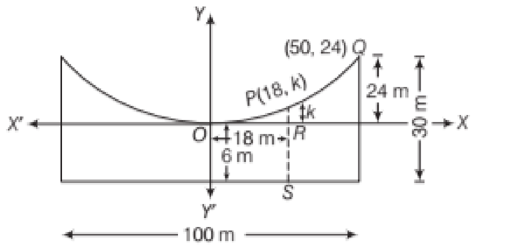
$\therefore \quad \left(\dfrac{k}{2}\right)^2 = \dfrac{5}{8} \times 2 \Rightarrow \dfrac{k^2}{4} = \dfrac{5}{4} \Rightarrow k = \sqrt{5} = 2.23$ m (approx.)

EXAMPLE |15| The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

Sol. Here, wires are vertical.

Let equation of the parabola be in the form

$$x^2 = 4ay \quad ...(i)$$



Focus is at the middle of the cable and shortest and longest vertical supports are 6 m and 30 m and roadway in 100 m long.

Clearly, the coordinates of $Q(50, 24)$ will satisfy Eq. (i).

$\therefore \ (50)^2 = 4a \times 24 \Rightarrow 2500 = 96a \Rightarrow a = \dfrac{2500}{96}$

Hence, from Eq. (i), $x^2 = 4 \times \dfrac{2500}{96} y$

$\Rightarrow \qquad x^2 = \dfrac{2500}{24} y$

Let $PR = k$ m

Then, point $P(18, k)$ will satisfy the equation of parabola.

$\therefore$ From Eq. (i), $(18)^2 = \dfrac{2500}{24} \times k$

$\Rightarrow \qquad 324 = \dfrac{2500}{24} k$

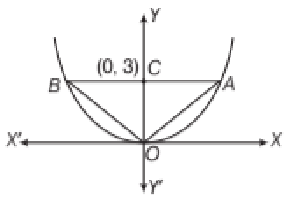
$\Rightarrow \qquad k = \dfrac{324 \times 24}{2500} = \dfrac{324 \times 6}{625} = \dfrac{1944}{625}$

$\Rightarrow \qquad k = 3.11$

$\therefore$ Required length $= 6 + k = 6 + 3.11 = 9.11$ m (approx.)

EXAMPLE |16| Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latusrectum. **[NCERT]**

Sol. We know that, latusrectum is a line perpendicular to the axis and passing through focus whose length is $4a$.



Given, $x^2 = 12y$, which is of the form $x^2 = 4ay$

$\therefore \qquad 4a = 12 =$ Length of ACB

Focus $C = (0, 3)$

$\therefore \quad$ Area of $\Delta OAB = \dfrac{1}{2} \times AB \times OC$

$= \dfrac{1}{2} \times 12 \times 3 = 6 \times 3$

$= 18$ sq units

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 The equation of a parabola is simplest, if the vertex is at the ...A... and the axis of symmetry is along the ...B... . Here, A and B respectively are

(a) origin, X-axis (b) origin, Y-axis
(c) Both (a) and (b) (d) None of these

2 The number of possible orientations of parabola is

(a) 1 (b) 2 (c) 3 (d) 4

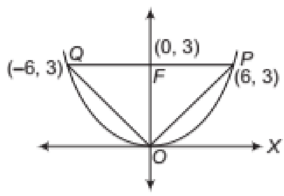
3 The eccentricity of parabola is always

(a) 0 (b) 1 (c) <1 (d) >1

4 When the axis of symmetry is along the X-axis the parabola opens to the

(a) right, if the coefficient of x is positive
(b) left, if the coefficient of x is negative
(c) Both (a) and (b)
(d) Neither (a) nor (b)

5 The area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latusrectum is **[NCERT Exemplar]**



(a) 12 sq units (b) 16 sq units
(c) 18 sq units (d) 24 sq units

VERY SHORT ANSWER Type Questions

6 Find the equation of the parabola whose focus is $(0, -3)$ and directrix is $y = 3$.

7 Find the equation of the parabola with vertex at the origin and focus at $(-2, 0)$.

8 Find the equation of the parabola with focus $(4, 0)$ and directrix is $x + 4 = 0$.

9 If the equation of the parabola is $x^2 = -8y$, then find the equation of directrix.

SHORT ANSWER Type I Questions

10 In each of the following questions, find the coordinates of the focus, the equation of the directrix axis of the parabola and length of the latusrectum. (Each point carries 2 Marks)

(i) $x^2 = -6y$

(ii) $x^2 = -16y$

(iii) $y^2 = 8x$

(iv) $y^2 = -16x$

(v) $x^2 = 36y$

11 Find the equation of the parabola which is symmetric about the Y-axis and passes through the point $(2, -3)$.

12 Find the equation of the parabola with vertex at the origin, the axis along the X-axis and passing through the point $(2, 3)$.

13 The focal distance of a point on the parabola $y^2 = 12x$ is 4. Find the abscissa of this point.

14 If the parabola $y^2 = 4ax$ passes through the point $(3, 2)$, find the length of its latusrectum.

15 Find the vertex and the directrix of the parabola $y^2 - 3x - 2y + 7 = 0$.

SHORT ANSWER Type II Questions

16 Find the equation of the parabola whose focus is $(1, -1)$ and vertex is $(2, 1)$.

17 Find the equation of the parabola whose vertex is $(6, -3)$ and directrix is $3x - 5y + 1 = 0$.

18 Find the vertex, focus, directrix and length of the latusrectum of the parabola $y^2 - 4y - 2x - 8 = 0$.

19 If the line $y = mx + 1$ is tangent to the parabola $y^2 = 4x$, then find the value of m.

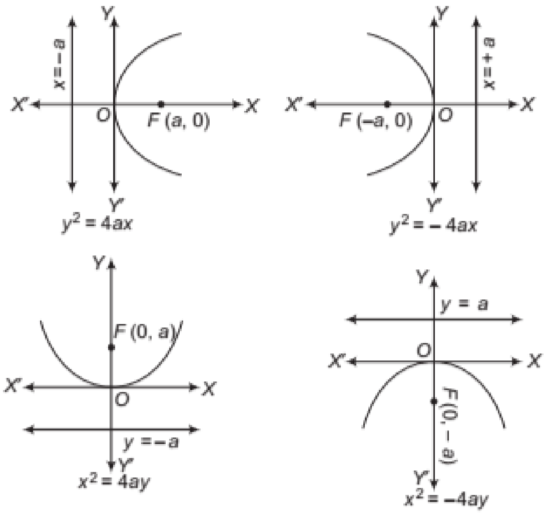
20 Find the vertex, focus, directrix and length of latusrectum of the parabola $2y^2 + 3y - 4x - 3 = 0$.

21 Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = -36y$ to the ends of the latusrectum.

HINTS & ANSWERS

1. (c) The equation of a parabola is simplest, if the vertex is at the origin and the axis of symmetry is along the X- axis or Y-axis.

2. (d) The number of possible orientations of parabola is four.



3. (b) The eccentricity of parabola is always 1.

4. (c) When the axis of symmetry is along the X-axis the parabola opens to the

(i) right, if the coefficient of x is positive.

(ii) left, if the coefficient of x is negative.

5. (c) From the figure, OPQ represents the triangle whose area is to be determined. The area of the triangle

$$= \frac{1}{2} PQ \times OF = \frac{1}{2}(12 \times 3) = 18$$

6. Required equation of parabola is $x^2 + (y+3)^2 = \frac{(y-3)^2}{(\sqrt{1})^2}$

Ans. $x^2 = -12y$

7. Equation of parabola is $y^2 = 4ax = 4(-2)x$.

Ans. $y^2 = -8x$

8. Equation of parabola is $(x-4)^2 + (y-0)^2 = \frac{(x+4)^2}{(\sqrt{1})^2}$.

Ans. $y^2 = 16x$

9. $y = 2$

10. (i) $\left(0, \frac{-3}{2}\right), y = \frac{3}{2}, x = 0, 6$

(ii) $(0, -4), y - 4 = 0, x = 0, 16$

(iii) $(2, 0), x = -2, x = 0, 8$

(iv) $(4, 0), x = -4, y = 0, 16$

(v) $(0, 9), y = -9, x = 0, 36$

11. Parabola is symmetrical about Y-axis and passes through $(2, -3)$.

So, equation of parabola is of the form $x^2 = -4ay$.

On putting $x = 2, y = -3$, we get $4 = -4a(-3) \Rightarrow a = \frac{1}{3}$

Ans. $3x^2 = -4y$

12. Equation of parabola, whose axis is X-axis and passing through the point (2, 3), will be of the form $y^2 = 4ax$.

On putting $x = 2, y = 3$, we get $9 = 4a(2) \Rightarrow a = \frac{9}{8}$

Ans. $2y^2 = 9x$

13. Given parabola is $y^2 = 12x$.

Here, $4a = 12 \Rightarrow a = 3$

$\therefore$ Focus : $F = (3, 0)$

Let $P(x, y)$ be any point on the parabola, then $PF = 4$

$\Rightarrow (x-3)^2 + (y-0)^2 = (4)^2$

$\Rightarrow x^2 + 9 - 6x + 12x = 16 \quad [\because y^2 = 12x]$

$\Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow x = 1 \quad [\because x \neq -7]$

Ans. 1

14. Parabola $y^2 = 4ax$ passes through the point (3, 2).

$\therefore (2)^2 = 4a(3) \Rightarrow a = \frac{1}{3}$

Then, length of latusrectum $= 4a = \frac{4}{3}$ **Ans.** 4/3

15. Given parabola is $y^2 - 3x - 2y + 7 = 0$

$\Rightarrow y^2 - 2y + 1 - 1 = 3x - 7$

$\Rightarrow (y-1)^2 = 3(x-2)$

which is of the form $Y^2 = 4aX$, where $Y = y - 1, X = x - 2$

Ans. $(2, 1), 4x - 5 = 0$

16. Solve as Example 4.

Ans. $4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$

17. Solve as Example 6.

Ans. $25x^2 + 9y^2 - 618x + 554y + 30xy + 4929 = 0$

18. Given equation is $y^2 - 4y - 2x - 8 = 0$

$(y-2)^2 = 2(x+6)$

For vertex, $y - 2 = 0 \Rightarrow y = 2$

$x + 6 = 0 \Rightarrow x = -6$

For focus, $x + 6 = \frac{1}{2}$

and $y - 2 = 0 \quad \left[\because a = \frac{1}{2}\right]$

For directrix, $x + 6 = -\frac{1}{2}$

and length of latusrectum $= 4a = 4\left(\frac{1}{2}\right) = 2$

Ans. $(6, -2), \left(-\frac{11}{2}, 2\right), 2x + 13 = 0, 2$

19. $m = 1$

20. Given equation of parabola is $2y^2 + 3y - 4x - 3 = 0$.

$\therefore 2\left[y^2 + 2 \cdot y \cdot \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right] = 4x + 3$

$\Rightarrow 2\left(y + \frac{3}{4}\right)^2 = 4x + 3 + \frac{9}{8}$

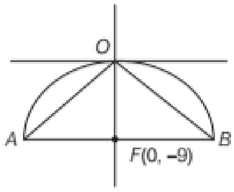
$\Rightarrow \left(y + \frac{3}{4}\right)^2 = 2\left(x + \frac{33}{32}\right)$

This is of the form $Y^2 = 4aX$, where $Y = y + \frac{3}{4}, X = x + \frac{33}{32}$

Ans. $\left(\frac{-33}{32}, \frac{-3}{4}\right), \left(\frac{-17}{32}, \frac{-3}{4}\right), x = \frac{-49}{32}, 2$

21. Given equation of parabola is $x^2 = -36y \Rightarrow x^2 = -4(9y)$

Here, $a = 9$



Focus $\equiv (0, -a) = (0, -9)$

Let AB be the latusrectum.

$\therefore y = -9$, then $x^2 = -36(-9)$

$\Rightarrow x = \pm 18$

So, the coordinates of A and B are $(-18, -9)$ and $(18, 9)$, respectively.

Area of $\Delta AOB = 2 \times$ Area of ΔOFB

$= 2 \times \left(\frac{1}{2} \times 18 \times 9\right)$

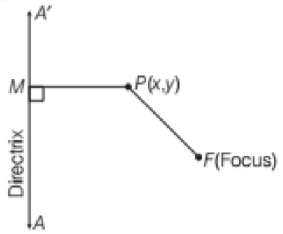
Ans. 162 sq units

|TOPIC 3|
Ellipse

An ellipse is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called **focus**) in the same plane to its distance from a fixed straight line (called **directrix**) is always constant, which is always less than unity.

The constant ratio is denoted by e and is called the eccentricity of the ellipse.

In the given figure, F is the focus, AA' is the directrix and P is any point on ellipse.

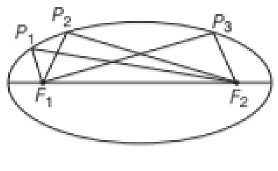


Then by definition,

$$\frac{PF}{PM} = e, 0 < e < 1$$

$$\Rightarrow \quad PF = e \cdot PM$$

In other words, we can say that, an ellipse is the set of all points in a plane, the sum of whose distance from two fixed points in the plane is a constant.



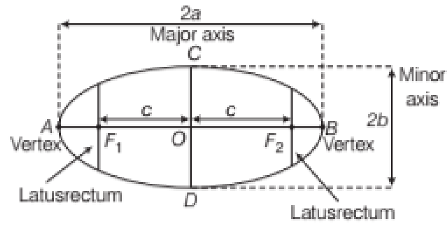
$$\therefore \; P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2 = P_3F_1 + P_3F_2 = \text{Constant}$$

Note

An ellipse is the locus of a point in a plane, which moves in such a way that the sum of its distance from two fixed point in the plane is a constant and this constant is greater than the distance between two fixed points.

Terms Related to an Ellipse

- **Focus** Two fixed points are called the foci (plural of focus) of the ellipse and denoted by F_1 and F_2. The distance between two foci F_1 and F_2 is $2c$.
- **Centre** The mid-point of the line segment joining the foci, is called centre of ellipse.
- **Major Axis** The line segment through the foci of the ellipse, is called the major axis. The length of major axis is denoted by $2a$.
- **Minor Axis** The line segment through the centre and perpendicular to the major axis is called minor axis. The length of minor axis is denoted by $2b$.



- **Vertices** The end points of the major axis, are called vertices of the ellipse.
- **Eccentricity** The eccentricity of ellipse is the ratio of the distance from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse. It is denoted by e.

Thus, $\quad e = \dfrac{c}{a} \Rightarrow c = ae$

Since, $\quad c < a \Rightarrow \dfrac{c}{a} < 1 \Rightarrow e < 1$

- **Latusrectum** Latusrectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse.

Thus, the length of latusrectum $= 2l = \dfrac{2b^2}{2}$

RELATION AMONG SEMI-MAJOR, SEMI-MINOR AXES AND DISTANCE OF FOCUS FROM THE CENTRE

Let a and b be the semi-major and semi-minor axes, respectively. Again, let c be the distance of the focus from the centre.

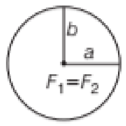
$\therefore$ Relation among a, b and c is given $c = \sqrt{a^2 - b^2}$

Note

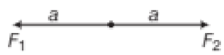
Special Cases of an Ellipse

In the equation $c^2 = a^2 - b^2$, if we keep a fixed and vary c from 0 to a, the resulting ellipse will vary in shape.

***Case* I** If $c = 0$, then $a = b$. Thus, ellipse becomes a circle.

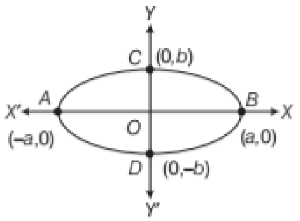


***Case* II** If $c = a$, then $b = 0$. Thus, ellipse reduce to line segment F_1F_2.



Standard Equation of an Ellipse

The standard equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$, having centre and major axis lies on X-axis and minor axis lies on Y-axis. It is also called horizontal ellipse.



Some important terms related to horizontal ellipse are

(i) Centre is $O\,(0, 0)$. (ii) Vertices are $(\pm a, 0)$.

(iii) Foci are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c=\sqrt{a^2-b^2}$.

(iv) Length of the major axis, $AB = 2a$

(v) Length of the minor axis, $CD = 2b$

(vi) Eccentricity, $e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$

(vii) Equation of major axis, $Y=0$

(viii) Equation of minor axis, $X=0$

(ix) Length of latusrectum $=\frac{2b^2}{a}$

(x) Vertices of latusrectum $=\left(\pm c, \pm\frac{b^2}{a}\right)$

(xi) Focal length $=2c$

Some important terms related to vertical ellipse are

(i) Centre is $O\,(0, 0)$.

(ii) Vertices are $C\,(0, a)$ and $D\,(0, -a)$.

(iii) Foci are $F_1(0, c)$ and $F_2(0, -c)$.

(iv) Length of the major axis, $CD = 2a$

(v) Length of the minor axis, $AB = 2b$

(vi) Eccentricity, $e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$

(vii) Equation of major axis, $X=0$

(viii) Equation of minor axis, $Y=0$

(ix) Length of latusrectum $=\frac{2b^2}{a}$

(x) Coordinates of latusrectum $=\left(\pm\frac{b^2}{a}, \pm c\right)$

Discussion

From above equation of an ellipse, it follows that for every point $P\,(x, y)$ on the ellipse, we have

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}\le 1 \text{ i.e. } x^2\le a^2$$

So, $$-a\le x\le a$$

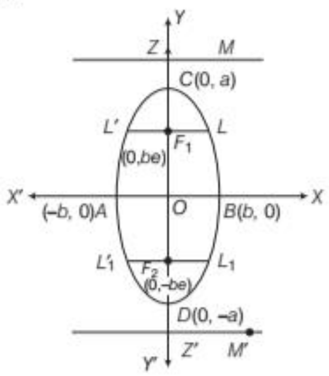
Therefore, the ellipse lie between the lines $x=-a$ and $x=a$ and touches these lines. Similarly, the ellipse lie between the lines $y=-b$ and $y=b$ and touches these lines.

Note

In equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if denominator of x^2 (i.e. a^2) is greater than the denominator of y^2, then it is a horizontal ellipse and resembles an egg lying on table and if denominator of y^2 (i.e. b^2) is greater than the denominator of x^2, then it is a vertical ellipse and resembles an egg standing on table.

Other form of an Ellipse

The another form of the equation of an ellipse is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\ a>b$, having centre at origin and major axis lies on Y-axis and minor axis lies on X-axis. It is also called vertical ellipse.



OBSERVATIONS FROM STANDARD EQUATION

From the standard equations of the ellipses, following observations can be explained through flow chart

Ellipse

↓

If we replace a point (x, y) either by $(x, -y)$ or $(-x, y)$ or $(-x, -y)$, then the equation of ellipse does not change. It means, it is symmetric with respect to both the coordinate axes.

↓

Foci are always lie on the major axis of an ellipse.

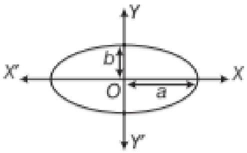
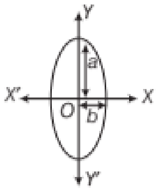
↓

- If the coefficient of x^2 has the larger denominator, then major axis is along X-axis.
- If the coefficient of y^2 has the larger denominator, then major axis is along Y-axis

Note

In this section, we study the standard equations of ellipses have centre at the origin and the major and minor axes at coordinate axes.

Comparison between Two Standard Ellipse

S.No.	Terms	Horizontal ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b>0$	Vertical ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\ a>b>0$
1.	Shape		
2.	Centre	$O(0, 0)$	$O(0, 0)$
3.	Vertices	$(\pm a, 0)$	$(0, \pm a)$
4.	Major axis	$2a$	$2a$
5.	Minor axis	$2b$	$2b$
6.	Value of c	$c=\sqrt{a^2-b^2}$	$c=\sqrt{a^2-b^2}$
7.	Equation of major axis	$y=0$	$x=0$
8.	Equation of minor axis	$x=0$	$y=0$
9.	Directrices	$x=\pm\frac{a^2}{c}$	$y=\pm\frac{a^2}{c}$
10.	Foci	$(\pm c, 0)$	$(0, \pm c)$
11.	Eccentricity	$e=\frac{c}{a}=\sqrt{1-\frac{b^2}{a^2}}$	$e=\frac{c}{a}=\sqrt{1-\frac{b^2}{a^2}}$
12.	Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
13.	Coordinates of latusrectum	$\left(\pm c, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{b^2}{a}, \pm c\right)$
14.	Focal radius	$\lvert F_1P\rvert=(a-ex_1)$ and $\lvert F_2P\rvert=(a+ex_1)$	$\lvert F_1P\rvert=(b-ey_1)$ and $\lvert F_2P\rvert=(b+ey_1)$
15.	Focal distance	$2c$	$2c$

Different Types of Questions Based on Ellipse

|TYPE I|

TO FIND THE DIFFERENT TERMS OF THE ELLIPSE WHEN EQUATION OF ELLIPSE IS GIVEN

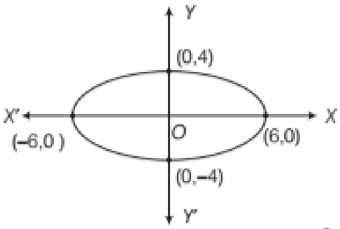
For finding the values of different terms, first we identify the equation of ellipse i.e. it is horizontal or vertical ellipse. Then, find the values of terms by using the formulae from the above chart.

EXAMPLE |1| Draw the shape of the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$ and find their major axis, minor axis, value of c, vertices, directrices, foci, eccentricity and length of latusrectum. **[NCERT]**

Sol. Given equation of ellipse is $\frac{x^2}{36}+\frac{y^2}{16}=1.$

Since, denominator of $\frac{x^2}{36}$ is greater than denominator of $\frac{y^2}{16}$.

$\therefore$ Major axis is along X-axis.



On comparing the above equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$

we get $a=6$ and $b=4$.

$\therefore$ Major axis $=2a=2\times6=12$

Minor axis $=2b=2\times4=8$

Value of $c=\sqrt{a^2-b^2}=\sqrt{(6)^2-(4)^2}$

$=\sqrt{36-16}=\sqrt{20}=2\sqrt5$

Vertices are $(6,0)$ and $(-6,0)$.

Eccentricity, $e=\sqrt{1-\left(\frac{b}{a}\right)^2}=\sqrt{1-\left(\frac{4}{6}\right)^2}$

$=\sqrt{\frac{36-16}{36}}=\sqrt{\frac{20}{36}}=\frac{\sqrt{20}}{6}$

Directrices are $x=\pm\frac{a}{e}=\pm\frac{6}{\sqrt{20}/6}=\pm\frac{36}{\sqrt{20}}$

and length of latusrectum $=\frac{2b^2}{a}=\frac{2(4)^2}{6}=\frac{16}{3}$

EXAMPLE |2| Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latusrectum of the ellipse $\frac{x^2}{4}+\frac{y^2}{25}=1$. **[NCERT]**

Sol. Given equation of ellipse is $\frac{x^2}{4}+\frac{y^2}{25}=1.$

Since, the denominator of $\frac{x^2}{4}$ is smaller than the denominator of $\frac{y^2}{25}$.

So, the major axis is along Y-axis.

On comparing the given equation with $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,$ we get

$b^2=4$ and $a^2=25$

$\Rightarrow \qquad b = 2$ and $a = 5$

Here, a and b are lengths, so we take only positive sign.

$\because \qquad c^2 = a^2 - b^2 = 25 - 4 = 21$

$\Rightarrow \qquad c = \sqrt{21}$

Here, the major axis is along Y-axis.

$\therefore \qquad$ Foci $= (0, \pm c) = (0, \pm \sqrt{21})$

Vertices $= (0, \pm a) = (0, \pm 5)$

Length of major axis $= 2a = 2 \times 5 = 10$

Length of minor axis $= 2b = 2 \times 2 = 4$

Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{21}}{5}$

Length of latusrectum $= \frac{2b^2}{a} = \frac{2 \times 4}{5} = \frac{8}{5}$

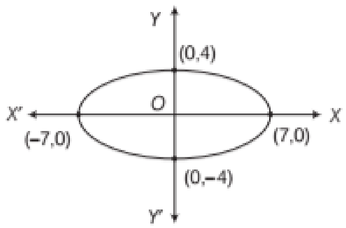
EXAMPLE |3| Draw the shape of ellipse $\frac{x^2}{49} + \frac{y^2}{16} = 1$ and find the

(i) major axis (ii) minor axis
(iii) value of c (iv) vertices
(v) directrices (vi) foci
(vii) eccentricity
(viii) length of latusrectum of given ellipse.

Sol. Given equation of ellipse is $\frac{x^2}{49} + \frac{y^2}{16} = 1$.

On comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a = 7, b = 4$



Here, $a > b$, so major axis is along X-axis.

(i) Major axis, $2a = 2 \times 7 = 14$

(ii) Minor axis, $2b = 2 \times 4 = 8$

(iii) $c = \sqrt{a^2 - b^2} = \sqrt{(7)^2 - (4)^2} = \sqrt{49 - 16} = \sqrt{33}$

(vi) Vertices, $(\pm a, 0) = (\pm 7, 0)$

(v) Directrices, $x = \pm \frac{a^2}{c} = \pm \frac{49}{\sqrt{33}}$

(vi) Foci, $(\pm c, 0) = (\pm \sqrt{33}, 0)$

(vii) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{33}}{7}$

(viii) Length of latusrectum,

$$\frac{2b^2}{a} = \frac{2 \times 16}{7} = \frac{32}{7}$$

|TYPE II|

TO FIND THE EQUATION OF AN ELLIPSE IN DIFFERENT CASES

***Case* I** When foci, eccentricity and directrix are given

If focus is $F(x_1, y_1)$, equation of directrix is $ax + by + c = 0$, eccentricity e and $P(x, y)$ is any point on the ellipse, then equation of ellipse is given by

$$(x - x_1)^2 + (y - y_1)^2 = \frac{e^2(ax + by + c)^2}{a^2 + b^2}$$

EXAMPLE |4| Find the equation of the ellipse, whose focus, directrix and eccentricity are respectively $(-1, 1)$, $x - y + 3 = 0$ and $\frac{1}{2}$.

Sol. Here, $F(-1, 1)$, equation of directrix $x - y + 3 = 0$ and $e = \frac{1}{2}$, then the equation of ellipse is

$$(x+1)^2 + (y-1)^2 = \left(\frac{1}{2}\right)^2 \frac{(x - y + 3)^2}{1^2 + 1^2}$$

$\Rightarrow \qquad (x+1)^2 + (y-1)^2 = \frac{(x - y + 3)^2}{8}$

$\Rightarrow \qquad 8[(x+1)^2 + (y-1)^2] = (x - y + 3)^2$

$\Rightarrow \qquad 8(x^2 + 2x + 1 + y^2 - 2y + 1)$

$= x^2 + y^2 + 9 - 2xy + 6x - 6y$

$\Rightarrow \qquad 7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

***Case* II** When vertices and foci are given

If the x-coordinate of vertices is zero, then required equation will be of vertical ellipse, otherwise horizontal ellipse. Now, compare the vertices and foci with standard values and simplify to get the values of a, e and b and put these values in required standard equation.

EXAMPLE |5| Find the equation of the ellipse whose axes are along the coordinate axes, vertices are $(\pm 5, 0)$ and foci at $(\pm 4, 0)$. **[NCERT]**

Sol. Given, vertices $\equiv (\pm 5, 0)$ and foci $\equiv (\pm 4, 0)$.

$\because$ y-coordinate of vertices are zero. So, let equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$

Now, $\qquad$ vertices $\equiv (\pm a, 0) = (\pm 5, 0)$

$\therefore \qquad a = 5$ and foci $\equiv (\pm ae, 0) = (\pm 4, 0)$

$\therefore \qquad ae = 4 \Rightarrow e = \frac{4}{a} = \frac{4}{5} \qquad [\because a = 5]$

Now, $b^2 = a^2(1 - e^2) = 25\left(1 - \frac{16}{25}\right) = 25\frac{(25 - 16)}{25} = 9$

Hence, the required equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

***Case* III When major (minor) axis and foci are given**
If major (minor) axis and foci are given, then find the values of a (or b) and c. After this, find the value of b (ro a) with the help of $c^2 = a^2 - b^2$. Now, put the values of a and b to get the required equation of an ellipse.

EXAMPLE |6| Find the equation of the ellipse, if foci are $(\pm 3, 0)$ and $a = 4$. **[NCERT]**

Sol. Given foci are on X-axis. So, the major axis will be along the X-axis. So, the equation of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a > b$$

Also, we have $a = 4$ and $c = 3$

$$\because \quad b^2 = a^2 - c^2 \Rightarrow b = \sqrt{a^2 - c^2}$$
$$= \sqrt{16 - 9} = \sqrt{7}$$

Hence, the equation of an ellipse is $\frac{x^2}{16} + \frac{y^2}{7} = 1.$

EXAMPLE |7| Find the equation of the ellipse, if length of major axis is 26 and foci $(\pm 5, 0)$. **[NCERT]**

Sol. We have, foci $(\pm 5, 0)$ which are on X-axis, so the equation of ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

Since, foci $\equiv (\pm c, 0) = (\pm 5, 0) \Rightarrow c = 5$

Length of major axis $= 2a = 26 \Rightarrow a = 13$

We know that, $c^2 = a^2 - b^2$

$$\Rightarrow \quad b = \sqrt{a^2 - c^2} = \sqrt{169 - 25} = \sqrt{144}$$

$$\therefore \quad b = 12$$

Hence, the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ i.e. } \frac{x^2}{169} + \frac{y^2}{144} = 1$$

EXAMPLE |8| Find the equation of the ellipse having length of minor axis $= 16$ and foci $(0, \pm 6)$. **[NCERT]**

Sol. Length of minor axis, $2b = 16 \Rightarrow b = 8$

Foci are $(0, \pm 6) \Rightarrow c = 6$

$$\therefore \quad a^2 = b^2 + c^2 = (8)^2 + (6)^2 = 64 + 36$$

$$\Rightarrow \quad a^2 = 100$$

Here, coefficient of x in foci is zero. So, major axis is on Y-axis. Then, equation of ellipse is

$$\frac{x^2}{64} + \frac{y^2}{100} = 1$$

***Case* IV When major axis and minor axis both are given**
If major and minor both axis are given, then find the value of a, b then put the value in the standard equation of ellipse.

EXAMPLE |9| Find the equation of the ellipse, if the ends of major axis are $(\pm 3, 0)$ and ends of minor axis are $(0, \pm 2)$. **[NCERT]**

Sol. Since, the vertices are on X-axis and therefore the equation is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$

Since, major and minor axes are $(\pm 3, 0)$ and $(0, \pm 2)$, respectively.

$$\therefore \quad a = 3 \text{ and } b = 2$$

Hence, the equation of an ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1.$

***Case* V When latusrectum and eccentricity are given**

EXAMPLE |10| Find the equation of ellipse, whose axes are along the axes of coordinates and centre at the origin and axes whose latusrectum $= 8$ and $e = \frac{1}{\sqrt{2}}$.

Sol. Let the equation of an ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

Given, latusrectum $= \frac{2b^2}{a} = 8$

$$\therefore \quad b^2 = 4a \quad \text{...(i)}$$

Also, $\quad e = \frac{1}{\sqrt{2}}; \ b^2 = a^2(1 - e^2)$

$$\therefore \quad 4a = a^2\left(1 - \frac{1}{2}\right) \quad \left[\text{put } e = \frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow \quad 4 = \frac{1}{2}a \Rightarrow a = 8 \text{ and } b^2 = 4a = 4 \times 8 = 32$$

Hence, the equation of ellipse is

$$\frac{x^2}{64} + \frac{y^2}{32} = 1 \quad [\text{put } a^2 = 64 \text{ and } b^2 = 32]$$

Note

In this question, major or minor axis is not clear, so we consider by default major axis as X-axis.

***Case* VI When other parts of an ellipse are given**
Sometimes other parts of an ellipse are given, then we find the values of a and b by using these parts or conditions and find the required equation of an ellipse.

EXAMPLE |11| Find the equation of ellipse, if it satisfies the condition $b = 3, c = 4$, centre at origin, foci on the X-axis.

Sol. Given, foci lies on X-axis. So, the equation of ellipse will be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b \quad \text{...(i)}$$

Also given that, $b = 3$ and $c = 4$

$\therefore \quad c^2 = a^2 - 3^2 \Rightarrow (4)^2 = a^2 - 9 \Rightarrow 16 = a^2 - 9$

$\Rightarrow \quad a^2 = 16 + 9 \Rightarrow a^2 = 25$

On putting the values of $a^2 = 25$

and $b^2 = 9$ in Eq. (i), we get $\frac{x^2}{25} + \frac{y^2}{9} = 1$

EXAMPLE |12| Find the equation of the ellipse, whose distance between directrices is 5 and distance between foci is 4.

Sol. Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

Then, equation of directrices are

$$x = \pm \frac{a^2}{c} = \pm \frac{a^2}{ae} = \pm \frac{a}{e}$$

i.e. $\quad x = \frac{a}{e}$ and $x = -\frac{a}{e}$

$\therefore$ Distance between directrices $= \frac{2a}{e} = 5$

$\therefore \quad e = \frac{2a}{5}$

Also, its foci are $(\pm c, 0)$ or $(\pm ae, 0)$.

$\therefore$ Distance between foci $= 2ae = 4 \Rightarrow ae = 2$

$\Rightarrow \quad a\left(\frac{2a}{5}\right) = 2 \Rightarrow a^2 = 5 \quad \left[\text{put } e = \frac{2a}{5}\right]$

$\therefore \quad e = \frac{2\sqrt{5}}{5} = \frac{2}{\sqrt{5}}$ and $b^2 = a^2(1 - e^2)$

$\Rightarrow \quad b^2 = 5\left(1 - \frac{4}{5}\right) = 5 \times \frac{1}{5} \quad [\text{put } a^2 = 5]$

$\therefore \quad b^2 = 1$

Hence, the equation of ellipse is

$$\frac{x^2}{5} + \frac{y^2}{1} = 1 \quad [\text{put } a^2 = 5 \text{ and } b^2 = 1]$$

$\Rightarrow \quad x^2 + 5y^2 = 5$

EXAMPLE |13| Find the equation of the ellipse, whose foci are $(\pm 3, 0)$ and passing through $(4, 1)$.

Sol. We have, foci of ellipse at $(\pm 3, 0)$ which are on X-axis.

Therefore, equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b \quad \text{...(i)}$$

Its foci are $(\pm c, 0) = (\pm 3, 0)$

$\therefore \quad c = 3$

Now, $\quad c^2 = a^2 - b^2$

$\Rightarrow \quad 9 = a^2 - b^2 \quad [\because ae = 3]$ ...(ii)

Since, Eq. (i) passes through $(4, 1)$.

$\therefore \quad \frac{16}{a^2} + \frac{1}{b^2} = 1 \Rightarrow \frac{16}{9 + b^2} + \frac{1}{b^2} = 1 \quad$ [from Eq. (ii)]

$\Rightarrow \quad 16b^2 + 9 + b^2 = b^2(9 + b^2)$

$\Rightarrow \quad 17b^2 + 9 = 9b^2 + b^4$

$\Rightarrow \quad b^4 - 8b^2 - 9 = 0$

$\Rightarrow \quad (b^2 - 9)(b^2 + 1) = 0 \Rightarrow b^2 = 9$ or -1

But $\quad b^2 \neq -1 \Rightarrow b^2 = 1$

From Eq. (ii), we get $\quad a^2 = 9 + b^2 \Rightarrow a^2 = 9 + 9$

$\Rightarrow \quad a^2 = 18$

On putting the values of a^2 and b^2 in Eq. (i), we get

$$\frac{x^2}{18} + \frac{y^2}{9} = 1 \Rightarrow x^2 + 2y^2 = 18$$

EXAMPLE |14| Find the equation of an ellipse whose foci are $(\pm 4, 0)$ and the eccentricity is $\frac{1}{3}$.

Sol. We have, foci $(\pm 4, 0)$ which lie on X-axis, so the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

Since, $\quad$ foci $\equiv (\pm c, 0) = (\pm 4, 0) \Rightarrow c = 4$

We know that, $e = \frac{c}{a} \Rightarrow a = \frac{c}{e} = \frac{4}{1/3} = 12 \quad \left[\because e = \frac{1}{3}\right]$

$\therefore \quad a = 12$

Now, $\quad c^2 = a^2 - b^2 \Rightarrow (4)^2 = (12)^2 - (b)^2$

$\Rightarrow \quad b^2 = 144 - 16 = 128$

$\therefore \quad a^2 = (12)^2 = 144$ and $b^2 = 128$

Hence, the equation of the ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$.

EXAMPLE |15| Find the equation of ellipse with centre at the origin, major axis on the Y-axis and passing through the points $(3, 2)$ and $(1, 6)$. **[NCERT]**

Sol. We have, major axis of the ellipse lies on Y-axis. So, the equation of ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \ a > b$$

Since, this ellipse passes through the points $(3, 2)$ and $(1, 6)$.

$\therefore \quad \frac{9}{b^2} + \frac{4}{a^2} = 1 \quad$ ...(i)

and $\quad \frac{1}{b^2} + \frac{36}{a^2} = 1 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get $a^2 = 40, b^2 = 10$

Hence, the equation of ellipse is $\frac{x^2}{10} + \frac{y^2}{40} = 1$.

EXAMPLE |16| Find the equation of the ellipse whose centre lies at origin, major axis lies on the X-axis, the eccentricity is $\frac{2}{3}$ and the length of the latusrectum is 5 units.

Sol. We have, major axis of ellipse lies on the X-axis, so the equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$$

Length of its latusrectum $=\frac{2b^2}{a}=5$

$$\Rightarrow \quad \frac{2a^2(1-e^2)}{a}=5 \qquad [\because b^2=a^2(1-e^2)]$$

$$\Rightarrow \quad 2a\left(1-\frac{4}{9}\right)=5 \qquad [\because e=2/3,\text{ given}]$$

$$\Rightarrow \quad a=\frac{9}{2}\text{ and } b^2=\frac{5}{2}a=\frac{5}{2}\times\frac{9}{2}=\frac{45}{4}$$

$$\therefore \quad a^2=\left(\frac{9}{2}\right)^2=\frac{81}{4} \Rightarrow b^2=\frac{45}{4}$$

Hence, the equation of ellipse is

$$\frac{4x^2}{81}+\frac{4y^2}{45}=1$$

i.e. $$20x^2+36y^2=405$$

EXAMPLE |17| Find the equation of the ellipse referred to its axes as the axes of coordinates with latusrectum of length 4 and distance between foci $4\sqrt{2}$.

Sol. Let the equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$

$$\text{Length of latusrectum}=\frac{2b^2}{a}=4 \qquad \text{[given]}$$

$$\Rightarrow \quad b^2=2a \qquad \text{...(i)}$$

$$\text{Distance between the foci}=2ae=4\sqrt{2} \qquad \text{[given]}$$

$$\therefore \quad ae=2\sqrt{2} \qquad \text{...(ii)}$$

$$\because \quad b^2=a^2(1-e^2)$$

$$\therefore \quad 2a=a^2-a^2e^2 \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad 2a=a^2-(2\sqrt{2})^2 \qquad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad 2a=a^2-8 \Rightarrow a^2-2a-8=0$$

$$\Rightarrow \quad (a-4)(a+2)=0 \Rightarrow a=4 \text{ or } -2$$

But a cannot be negative.

$$\therefore \quad a=4$$

From Eq. (i), we get $b^2=2a \Rightarrow b^2=2\times4=8$

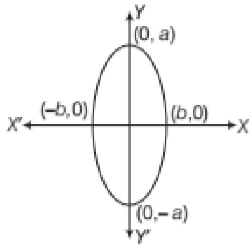
Hence, equation of ellipse is

$$\frac{x^2}{16}+\frac{y^2}{8}=1 \Rightarrow x^2+2y^2=16$$

EXAMPLE |18| Find the equation of the ellipse, if the centre is at (0, 0), major axis on the Y-axis and passing through the points (3, 2) and (1, 6). **[NCERT]**

Sol. Since, major axis is along Y-axis, so equation of ellipse is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\ a>b \qquad \text{...(i)}$$



Thus, the equation of ellipse is passing through the points (3, 2) and (1, 6). So, the point (3, 2) lies on Eq. (i), which gives

$$\frac{9}{b^2}+\frac{4}{a^2}=1 \qquad \text{...(ii)}$$

Also, the point (1, 6) lies on Eq. (i).

$$\therefore \quad \frac{1}{b^2}+\frac{36}{a^2}=1 \Rightarrow \frac{9}{b^2}+\frac{324}{a^2}=9 \qquad \text{...(iii)}$$

On subtracting Eq. (ii) from Eq. (iii), we get

$$\frac{320}{a^2}=8 \Rightarrow a^2=\frac{320}{8}=40$$

On putting the value of a^2 in Eq. (ii), we get

$$\frac{9}{b^2}+\frac{4}{40}=1 \Rightarrow \frac{9}{b^2}=1-\frac{1}{10}=\frac{9}{10} \Rightarrow b^2=10$$

On putting the values of a and b in Eq. (i), we get

$$\frac{x^2}{10}+\frac{y^2}{40}=1$$

EXAMPLE |19| Find the equation of the ellipse, whose major axis is on X-axis and passes through (4, 3) and (6, 2). **[NCERT]**

Sol. Let the major axis be along X-axis, then the equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b \qquad \text{...(i)}$$

Since, the equation of ellipse is passing through the points (4, 3) and (6, 2). So, the point (4, 3) lies on it.

$$\therefore \quad \frac{16}{a^2}+\frac{9}{b^2}=1 \qquad \text{...(ii)}$$

Also, the point (6, 2) lies on it.

$$\therefore \quad \frac{36}{a^2}+\frac{4}{b^2}=1 \qquad \text{...(iii)}$$

On multiplying Eq. (ii) by 4 and Eq. (iii) by 9 and then subtracting Eq. (ii) from Eq. (iii), we get

$$\frac{324}{a^2}-\frac{64}{a^2}=9-4$$

$$\Rightarrow \quad \frac{260}{a^2} = 5 \Rightarrow a^2 = \frac{260}{5} = 52 \Rightarrow a^2 = 52$$

On putting the value of a^2 in Eq. (ii), we get

$$\frac{16}{52} + \frac{9}{b^2} = 1 \Rightarrow \frac{9}{b^2} = 1 - \frac{16}{52} = \frac{36}{52} \Rightarrow b^2 = 13$$

Hence, the equation of ellipse is

$$\frac{x^2}{52} + \frac{y^2}{13} = 1 \quad \text{[put } a^2 = 52 \text{ and } b^2 = 13 \text{ in Eq. (i)]}$$

EXAMPLE |20| Find the equation of ellipse having major and minor axes along X and Y-axes respectively, the distance between whose foci is 8 units and the distance between the directrices is 18 units.

Sol. We have, distance between foci $= 8$

$$\therefore \quad 2ae = 8 \quad \text{...(i)}$$

and distance between directrices $= 18$

$$\therefore \quad \frac{2a}{e} = 18 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$a = 6 \text{ and } e = \frac{2}{3}$$

We know that,

$$(ae)^2 = a^2 - b^2 \Rightarrow (4)^2 = (6)^2 - b^2$$

$$\Rightarrow \quad b^2 = 36 - 16 = 20$$

Hence, equation of ellipse is

$$\frac{x^2}{36} + \frac{y^2}{20} = 1$$

EXAMPLE |21| Find the equation of the ellipse whose foci are $(2, 3)$, $(-2, 3)$ and whose length of semi-minor axis is $\sqrt{5}$.

Sol. We have, $F \equiv (2, 3)$,

$$F' \equiv (-2, 3), b = \sqrt{5}$$

Now, $FF' = \sqrt{(-2-2)^2 + (3-3)^2} = \sqrt{4^2} = 2$

$$\Rightarrow \quad 2ae = 4 \Rightarrow ae = 2$$

We know that, $(ae)^2 = a^2 - b^2$

$$\Rightarrow \quad 4 = a^2 - 5 \Rightarrow a^2 = 9 \Rightarrow a = 3$$

Let $P(x, y)$ be any point on the ellipse, then by another definition of ellipse,

$$PF + PF' = 2a$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} + \sqrt{(x+2)^2 + (y-3)^2} = 6$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} = 6 - \sqrt{(x+2)^2 + (y-3)^2}$$

On squaring both sides, we get

$$(x-2)^2 + (y-3)^2 = 36 + (x+2)^2 + (y-3)^2 - 12\sqrt{(x+2)^2 + (y-3)^2}$$

$$\Rightarrow (x+2)^2 - (x-2)^2 + 36 = 12\sqrt{(x+2)^2 + (y-3)^2}$$

$$\Rightarrow x^2 + 4x + 4 - (x^2 - 4x + 4) + 36 = 12\sqrt{(x+2)^2 + (y-3)^2}$$

$$\Rightarrow 8x + 36 = 12\sqrt{(x+2)^2 + (y-3)^2}$$

$$\Rightarrow 2x + 9 = 3\sqrt{(x+2)^2 + (y-3)^2} \quad \text{[divide both sides by 4]}$$

$$\Rightarrow (2x+9)^2 = 9[(x+2)^2 + (y-3)^2] \quad \text{[squaring both sides]}$$

$$\Rightarrow \quad 4x^2 + 36x + 81 = 9[x^2 + 4x + 4 + y^2 - 6y + 9]$$

$$\Rightarrow \quad 5x^2 + 9y^2 - 54y + 36 = 0$$

|TYPE III|

PROBLEMS BASED ON FINDING ECCENTRICITY IN DIFFERENT CONDITIONS

EXAMPLE |22| If the latusrectum of an ellipse is equal to half of minor axis, then find its eccentricity.

[NCERT Exemplar]

Sol. We know that, latusrectum of an ellipse is equal to the half of minor axis.

$$\therefore \quad \text{Length of latusrectum} = \frac{2b^2}{a}$$

and length of minor axis $= 2b$

According to the condition, $\frac{2b^2}{a} = \frac{1}{2}(2b) \Rightarrow \frac{b}{a} = \frac{1}{2}$

We know that, $e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow e = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

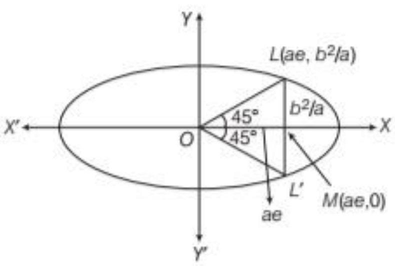
Hence, the eccentricity of ellipse is $\frac{\sqrt{3}}{2}$.

EXAMPLE |23| If a latusrectum of an ellipse subtends a right angle at the centre of the ellipse, then write the eccentricity of the ellipse.

Sol. Let equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \; a > b$$

Since, latusrectun LL' subtends an angle of $90°$ to the centre.



$$\therefore \quad \angle LOL' = 90°$$

Since, OM is the median of $\Delta OLL'$.

$$\therefore \quad \angle LOM = \angle L'OM = 45°$$

In right angled ΔOML,

$$\tan 45° = \frac{b^2/a}{ae} \Rightarrow b^2 = a^2 e \qquad [\because \tan 45° = 1]$$

$$\therefore \quad e^2 = 1 - \frac{b^2}{a^2} \Rightarrow e^2 = 1 - \frac{a^2 e}{a^2}$$

$$\Rightarrow \quad e^2 + e - 1 = 0$$

$$\therefore \quad e = \frac{-1 \pm \sqrt{1^2 + 4 \times 1}}{2 \times 1} = \frac{-1 \pm \sqrt{5}}{2}$$

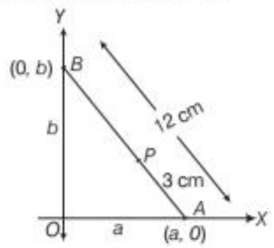
$$\therefore \quad e = \frac{-1 + \sqrt{5}}{2} \qquad [\because 0 < e < 1]$$

|TYPE IV|

PROBLEMS BASED ON APPLICATION OF ELLIPSE

EXAMPLE |24| A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the X-axis. **[NCERT]**

Sol. Let AB be the rod of length 12 cm.



Let $\quad OA = a$ and $OB = b$

Using Pythagoras theorem in ΔOAB,

$$a^2 + b^2 = (12)^2 \Rightarrow a^2 + b^2 = 144 \qquad \ldots(i)$$

Let $AP = 3$ cm, then $PB = AB - AP = 12 - 3 = 9$ cm

$\therefore \quad AP : PB = 3 : 9 = 1 : 3$

Let $P = (h, k)$ for which locus to be found.

If $P(h, k)$ divides the points A and B internally in the ratio $m : n$, then

$$h = \frac{mx_2 + nx_1}{m + n} \text{ and } k = \frac{my_2 + ny_1}{m + n}$$

$$\therefore \quad h = \frac{1 \times 0 + 3 \times a}{1 + 3} \Rightarrow h = \frac{3a}{4}$$

$$\Rightarrow \quad a = \frac{4h}{3} \text{ and } k = \frac{1 \times b + 3 \times 0}{1 + 3}$$

$$\Rightarrow \quad k = \frac{b}{4} \Rightarrow b = 4k$$

Put the values of $a = \frac{4h}{3}$ and $b = 4k$ in Eq. (i), we get

$$\left(\frac{4h}{3}\right)^2 + (4k)^2 = 144 \Rightarrow \frac{16h^2}{9} + 16k^2 = 144$$

$$\Rightarrow \quad \frac{h^2}{9} + k^2 = 9 \Rightarrow \frac{h^2}{81} + \frac{k^2}{9} = 1$$

Hence, the locus of a point $P(h, k)$ is $\frac{x^2}{81} + \frac{y^2}{9} = 1$.

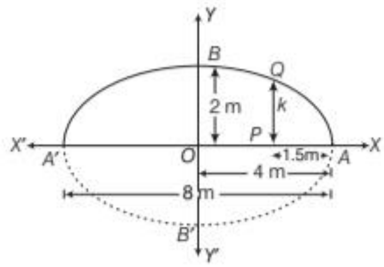
EXAMPLE |25| An arch is the form of a semi-ellipse. It is 8 m wide and 2 m high of the centre. Find the height of the arch at a point 1.5 m from one end. **[NCERT]**

Sol. Clearly, equation of ellipse takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots(i)$$

Here, it is given that $2a = 8$ and $b = 2$

$$\Rightarrow \quad a = 4 \text{ and } b = 2$$



Put the values of a and b in Eq. (i), we get

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

Given, $\quad AP = 1.5$ m

$\Rightarrow \quad OP = OA - AP = 4 - 1.5$

$\Rightarrow \quad OP = 2.5$ m

Let $\quad PQ = k$

Then, the coordinate $Q(2.5, k)$ will satisfy the equation of ellipse.

$$\text{i.e.} \quad \frac{(2.5)^2}{16} + \frac{k^2}{4} = 1 \Rightarrow \frac{6.25}{16} + \frac{k^2}{4} = 1$$

$$\Rightarrow \quad \frac{k^2}{4} = \frac{1}{1} - \frac{6.25}{16} = \frac{16 - 6.25}{16}$$

$$\Rightarrow \quad \frac{k^2}{4} = \frac{9.75}{16} \Rightarrow k^2 = \frac{9.75}{4}$$

$$\Rightarrow \quad k^2 = 2.4375$$

$$\therefore \quad k = 1.56 \text{ m (approx.)}$$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 If 'e' is the eccentricity and 'a' is the length of semi-major axis, then the focus is at a distance ...A... from the centre. Here, A stands for

(a) ae (b) $\frac{e}{a}$

(c) $\frac{a}{e}$ (d) None of these

2 If the centre of ellipse is at origin and major axis is along Y-axis, then the equation of ellipse is

(a) $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ (where $b>a$)

(b) $\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$ (where $b>a$)

(c) $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ (where $a>b$)

(d) $\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$ (where $a>b$)

3 The standard equations of ellipse have ...A... at the origin and the major and minor axes along ...B... . Here, A and B respectively are

(a) focus, coordinate axes
(b) centre, coordinate axes
(c) Both (a) and (b)
(d) Neither (a) nor (b)

4 If the foci and vertices of an ellipse be $(\pm 1, 0)$ and $(\pm 2, 0)$ respectively, then the minor axis of the ellipse is

(a) $2\sqrt{5}$ (b) 2 (c) 4 (d) $2\sqrt{3}$

5 The length of the latusrectum of an ellipse is $\frac{1}{3}$ of the major axis. Its eccentricity is

(a) $\frac{2}{3}$ (b) $\sqrt{\frac{2}{3}}$

(c) $\frac{5\times 4\times 3}{7^3}$ (d) $\left(\frac{3}{4}\right)^4$

VERY SHORT ANSWER Type Questions

6 If P is a point on the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ whose foci are S and S', then find the value of $PS + PS'$.

7 Find the distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$.

8 Find the eccentricity of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

9 Find the length of the latusrectum of the ellipse $3x^2 + y^2 = 12$.

10 If the latusrectum of an ellipse is equal to the half of its major axis, then find its eccentricity.

11 Find the equation of the ellipse, if the ends of major axis are $(\pm 2, 0)$ and ends of minor axis are $(0, \pm 7)$.

12 Find the equation of ellipse, if it satisfies the condition $b = 4$, $c = 9$, centre at origin , foci on the X-axis.

13 Find the equation of ellipse, if it satisfies the condition $b = 3$, $c = 8$, centre at origin, foci an the Y-axis.

SHORT ANSWER Type I Questions

14 Find the equation of the ellipse, whose foci $(0, \pm 5)$ and vertices $(0, \pm 13)$. [NCERT]

15 Find the equation of the ellipse with vertices at $(0, \pm 10)$ and $e = 4/5$.

16 Find the equation of the ellipse, if foci are $(\pm 5, 0)$ and $a = 6$.

17 Find the equation of the ellipse, if length of major axis is 22 and foci $(\pm 3, 0)$.

18 Find the equation of the ellipse, where distance between directices is 8 and distance between foci is 2.

19 Find the equation of the ellipse referred to its axes as the axes of coordinates with latusrectum of length 8 and distane between foci $6\sqrt{3}$.

20 Find the equation of the ellipse, whose axes along coordinates axes, passing through (4, 3) and (–1, 4). [NCERT]

21 If the eccentricity of an ellipse is 5/8 and distance between its foci is 10, then find latusrectum of the ellipse. [NCERT Exemplar]

22 Find the equation of the ellipse having foci $(0, \pm 1)$ and length of whose minor axis is unity.

23 Find the equation of the ellipse with foci at $(\pm 5, 0)$ and $x=\frac{36}{5}$ as one of the directrices.

24 Find the equation of ellipse whose eccentricity is 2/3 latusrectum is 5 and the centre is (0, 0).

SHORT ANSWER Type II Questions

25 Draw the shape of $\frac{x^2}{64}+\frac{y^2}{25}=1$ and find their vertices, major axis, minor axis, eccentricity, foci and length of latusrectum.

26 Draw the shape of $\frac{x^2}{100}+\frac{y^2}{400}=1$ and find their vertices major axis, minor axis, eccentricity foci, and length of latusrectum.

27 Find the equation of the ellipse, whose foci are $(\pm 4, 0)$ and passing through $(3, 2)$.

28 Find the equation of the ellipse whose centre is at origin and the X-axis, the major axis, which passes through the points $(-3, 1)$ and $(2, -2)$.

29 Find the equation of the ellipse whose focus is $(1, -1)$, the directrix is the line $x - y - 3 = 0$ and eccentricity is 1/2. **[NCERT Exemplar]**

30 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latusrectum of ellipse $3x^2 + 2y^2 - 6 = 0$.

31 Find the equation of the set of all points, the sum of whose distance from the points (3, 0) and (9, 0) is 12.

32 Find the equation of the ellipse passing through (6, 4), foci is on Y-axis, centres at the origin having eccentricity 3/4.

33 An arch is in the form of a semi-ellipse. It is 10 m wide and 3 m high of the centre. Find the height of the arch at a point 2 m from one end.

HINTS & ANSWERS

1. (a) The coordinates of foci are $(\pm, ae, 0)$. Therefore, its distance from the centre is (0, 0).

2. (c) If $a > b$ and major axis is along X-axis, then the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

   Similarly, if $a > b$ and major axis is along Y-axis, then the equation of ellipse is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

3. (b) The standard equations of ellipse have **centre** at the origin and the major and minor axis along **coordinate axes**.

4. (d) Given that, $ae = 1$, $a = 2$, $e = \frac{1}{2} \Rightarrow b = \sqrt{4\left(1-\frac{1}{4}\right)} = \sqrt{3}$

   Hence, minor axis $= 2\sqrt{3}$

5. (b) Latusrectum = 1/3 (major axis)

   $$\Rightarrow \quad \frac{2b^2}{a} = \frac{2a}{3} \Rightarrow a^2 = 3b^2 = 3a^2(1-e^2)$$

   $$\Rightarrow \quad e = \sqrt{\frac{2}{3}}$$

6. $PS + PS' = 2a$ **Ans.** 10

7. Distance between directrices $= \frac{2a}{e}$

   Now, $e = \sqrt{1-\frac{20}{36}} = \frac{4}{6}$ **Ans.** 18

8. $e = \sqrt{1-\frac{b^2}{a^2}}; a^2 = 9, b^2 = 4$ **Ans.** $e = \frac{\sqrt{5}}{3}$

9. $\frac{x^2}{4}+\frac{y^2}{12}=1$

   Length of latusrectum $= \frac{2b^2}{a}$, where $a^2 > b^2$ **Ans.** $\frac{4\sqrt{3}}{3}$

10. $\frac{2b}{a} = \frac{1}{2}(2\times a) \Rightarrow b = \frac{a^2}{2}$

    $\because e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{a^2/2}{a^2}}$ **Ans.** $\frac{1}{\sqrt{2}}$

11. Solve as Example 9. **Ans.** $\frac{x^2}{4}+\frac{y^2}{49}=1$

12. Solve as Example 11. **Ans.** $\frac{x^2}{97}+\frac{y^2}{16}=1$

13. Let the equation of an ellipse be $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$.

    $\because \quad c^2 = a^2 - b^2$ **Ans.** $\frac{y^2}{73}+\frac{x^2}{9}=1$

14. $(0, \pm c) = (0, \pm 5)$, $(0, \pm a) = (0, \pm 13) \Rightarrow c = 5$ and $a = 13$

    $\because b^2 = a^2 - c^2 = 169 - 25 \Rightarrow b^2 = 144$ **Ans.** $\frac{x^2}{144}+\frac{y^2}{169}=1$

15. Here, $(0, \pm a) = (a, \pm 10)$ and $e = \frac{4}{5}$

    $\because e^2 = 1-\frac{b^2}{a^2} \Rightarrow \frac{16}{25} = 1-\frac{b^2}{100} \Rightarrow \frac{b^2}{100} = \frac{9}{25} \Rightarrow b^2 = 36$

    $\therefore \quad \frac{x^2}{36}+\frac{y^2}{100}=1$ **Ans.** $25x^2 + 9y^2 = 900$

16. Solve as Example 6. **Ans.** $\frac{x^2}{36}+\frac{y^2}{11}=1$

17. Solve as Example 7. **Ans.** $\frac{x^2}{121}+\frac{y^2}{112}=1$

18. Solve as Example 12. **Ans.** $\frac{x^2}{4}+\frac{y^2}{3}=1$

19. Solve as Example 17. **Ans.** $\dfrac{x^2}{35+4\sqrt{31}}+\dfrac{y^2}{8+8\sqrt{31}}=1$

20. Let equation of ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, a>b$

$\therefore\ \dfrac{16}{a^2}+\dfrac{9}{b^2}=1$ and $\dfrac{1}{a^2}+\dfrac{16}{b^2}=1 \Rightarrow b^2=\dfrac{247}{15}$ and $a^2=\dfrac{247}{7}$

Ans. $7x^2+15y^2=247$

21. $e=\dfrac{5}{8}$, $2ae=10 \Rightarrow a=\dfrac{5}{5/8}=8\ \therefore\ e=\sqrt{1-\dfrac{b^2}{64}}$

$\dfrac{25}{64}=\dfrac{64-b^2}{64} \Rightarrow b^2=39.$ Latusrectum $=\dfrac{2b^2}{a}$ **Ans.** $\dfrac{39}{4}$

22. $20x^2+4y^2=5$

23. $\dfrac{x^2}{36}+\dfrac{y^2}{11}=1$

24. $\dfrac{4x^2}{81}+\dfrac{4y^2}{45}=1$

25. Solve as Example 1.

Ans. Vertices $=(\pm 8, 0)$, Major axis $=16$,

Minor axis $=10$, Eccentricity $=\dfrac{\sqrt{39}}{8}$

Foci $=(\pm\sqrt{39}, 0)$, Latusrectum $=\dfrac{25}{4}$

26. Solve as Example 2.

Ans. Vertices $=(0, \pm 20)$

Major axis $=12$, Minor axis $=20$, $e=\dfrac{\sqrt{3}}{2}$

Foci $=(0, \pm 10\sqrt{3})$, Latusrectum $=10$

27. Solve as Example 13. **Ans.** $\dfrac{x^2}{\dfrac{29+\sqrt{265}}{2}}+\dfrac{y^2}{\dfrac{-3+\sqrt{265}}{2}}=1$

28. $3x^2+5y^2=32$

29. $\sqrt{(x-1)^2+(y+1)^2}=\dfrac{1}{2}\cdot\dfrac{|x-y-3|}{\sqrt{1^2+1^2}}$

$\Rightarrow 8[(x^2+1-2x)+(y^2+1+2y)]$

$=x^2+y^2+9-2xy+6y-6x$

Ans. $7x^2+7y^2+2xy+10x-10y+7=0$

30. $(0, \pm 1)$, $(0, \pm\sqrt{3})$, $2\sqrt{3}$, $2\sqrt{2}$, $\dfrac{1}{\sqrt{3}}$, $\dfrac{4}{\sqrt{3}}$

31. $3x^2+4y^2-36x=0$

32. $16x^2+7y^2=688$

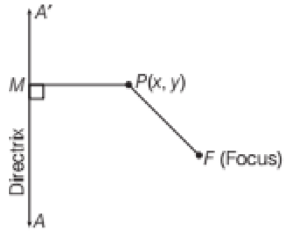
33. Solve as Example 25. **Ans.** 0.27

|TOPIC 4|
Hyperbola

A hyperbola, is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called **focus**) in the same plane to its distance from a fixed straight line (called **directrix**) is always constant, which is always greater than unity.

The constant ratio is denoted by e and is called the eccentricity of the hyperbola.

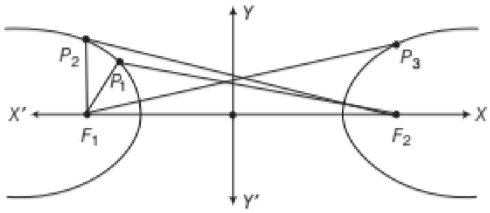
In the given figure, F is the focus, AA' is the directrix and P is any point on hyperbola.



Then by definition,

$$\frac{PF}{PM} = e, \text{ where } e > 1 \Rightarrow PF = e \cdot PM$$

In other words, we can say that, a hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.



$$\therefore \quad P_1F_2 - P_1F_1 = P_2F_2 - P_2F_1 = P_3F_1 - P_3F_2$$

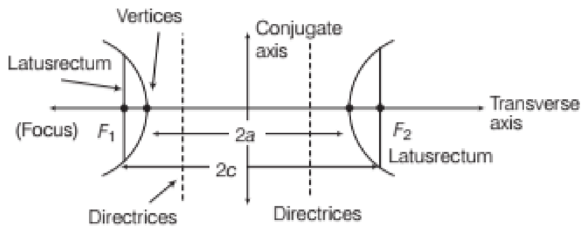
The term difference used in the definition means the distance to the farther point minus the distance to the closer point.

Note

A hyperbola is a locus of a point in a plane which moves in such a way that the difference of its distance from two fixed points in the plane is constant and this constant is less than the distance between two fixed points.

Terms Related to Hyperbola

- **Focus** The two fixed points are called the foci of the hyperbola and denoted by F_1 and F_2. The distance between two foci F_1 and F_2 is $2c$.
- **Centre** The mid-point of the line segment joining the foci, is called centre of hyberbola.
- **Transverse axis** The line through the foci, is called the transverse axis.



- **Conjugate axis** The line through the centre and perpendicular to the transverse axis, is called the conjugate axis.
- **Vertices** The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola. The distance between two vertices is 2a.
- **Eccentricity** Eccentricity of the hyperbola is the ratio of the distance of anyone focus from the centre and the distance of anyone vertex from the centre and it is denoted by e.

$$\therefore \quad e = \frac{c}{a} \text{ and } c > a$$

$$\Rightarrow \quad \frac{c}{a} > 1 \Rightarrow e > 1$$

- **Directrix** Directrix is a line perpendicular to the transverse axis and cuts it at a distance of $\frac{a^2}{c}$ from the centre.

i.e. $\quad x = \pm \frac{a^2}{c}$

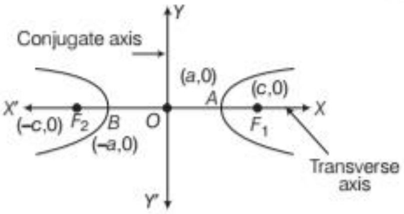
or $\quad y = \pm \frac{a^2}{c}$

- **Latusrectum** Latusrectum of a hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.

Thus, length of latusrectum $= 2l = \frac{2b^2}{a}$

STANDARD EQUATION OF HYPERBOLA

Standard equation of hyperbola is the form of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose X-axis as transverse axis and Y-axis as conjugate axis.

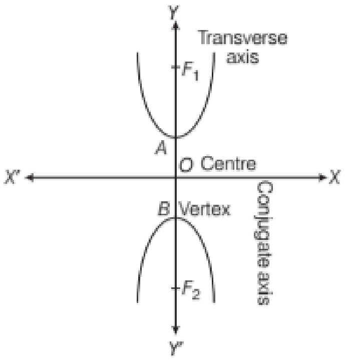


Some important terms related to standard hyperbola are

(i) Vertices are $A(a, 0)$ and $A'(-a, 0)$.
(ii) Centre is $O(0, 0)$.
(iii) Length of transverse axis is $2a$.
(iv) Length of conjugate axis is $2b$.
(v) Foci are $F_1(c, 0)$ and $F_2(-c, 0)$.
(vi) Directrices are $x = \frac{a^2}{c}$ and $x = -\frac{a^2}{c}$.
(vii) Eccentricity, $e = \sqrt{\frac{a^2 + b^2}{a^2}}$ or $b^2 = a^2(e^2 - 1)$
(viii) Length of latusrectum $= \frac{2b^2}{a}$
(ix) Coordinates of latusrectum $= \left(\pm ae, \pm \frac{b^2}{a}\right)$
(x) Focal length $= 2ae$

Conjugate Hyperbola

The equation of the hyperbola of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is called conjugate hyperbola, whose X-axis as conjugate axis and Y-axis as transverse axis.



Some important terms related to conjugate hyperbola are

(i) Vertices are $A(0, a)$ and $B(0, -a)$.
(ii) Centre is O (0, 0).
(iii) Transverse axis is $2a$.
(iv) Conjugate axis is $2b$.
(v) Foci are $(0, \pm c)$.
(vi) Directrices are $y = \frac{a^2}{c}$ and $y = \frac{-a^2}{c}$.
(vii) Eccentricity, $e = \sqrt{1 + \frac{b^2}{a^2}}$ or $b^2 = a^2(e^2 - 1)$
(viii) Length of latusrectum $= \frac{2b^2}{a}$
(ix) Coordinate of latusrectum $= \left(\pm \frac{b^2}{a}, \pm ae\right)$
(x) Focal length $= 2ae$

OBSERVATIONS FROM STANDARD EQUATION

From the standard equation of the hyperbola, following observations can be explained through flow chart

Hyperbola

If we replace a point (x, y) either by $(x, -y)$ or $(-x, y)$ or $(-x, -y)$, then the equation of hyperbola does not change. It means, it is symmetrical with respect to both the coordinate axes.

Foci are always lie on the transverse axis of the hyperbola.

If the denominator of x^2 gives the positive term, then transverse axis is along X-axis.	If the denominator of y^2 gives the positive term, then transverse axis is along Y-axis.

Discussion

From the standard equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We observe that, $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$

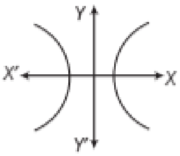
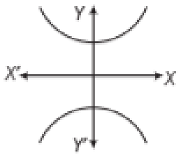
i.e. $\left|\frac{x}{a}\right| \geq 1 \Rightarrow |x| \geq |a| \Rightarrow x \leq -a$ and $x \geq a$

Hence, no portion of the curve lies between the lines $x = a$ and $x = -a$ (i.e. no real intercepts on the conjugate axis).

Note

In this section, we study the standard equations of hyperbolas which have transverse and conjugate axes as the coordinate axes and centre at the origin.

Comparison between Two Standard Hyperbola

S.No.	Terms	Hyperbola	Conjugate hyperbola
	Equation of hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
1.	Shape		
2.	Transverse axis	$2a$	$2a$
3.	Conjugate axis	$2b$	$2b$
4.	Value of c	$c=\sqrt{a^2+b^2}$	$c=\sqrt{a^2+b^2}$
5.	Vertices	$(\pm a, 0)$	$(0, \pm a)$
6.	Directrices	$x=\pm\frac{a^2}{c}$ or $\pm\frac{a}{e}$	$y=\pm\frac{a^2}{c}$ or $\pm\frac{a}{e}$
7.	Foci	$(\pm ae, 0)$ or $(\pm c, 0)$	$(0, \pm ae)$ or $(0, \pm c)$
8.	Eccentricity	$e=\sqrt{1+\frac{b^2}{a^2}}$ or $\frac{c}{a}$	$e=\sqrt{1+\frac{b^2}{a^2}}$ or $\frac{c}{a}$
9.	Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$

Different Types of Questions Based on Hyperbola

|TYPE I|

TO FIND DIFFERENT TERMS OF HYPERBOLA WHEN EQUATION OF HYPERBOLA IS GIVEN

For finding the different parts of a given hyperbola, first we check that given hyperbola is of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ or } \frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

Then, we find the terms of hyperbola by using the formulae from the above chart.

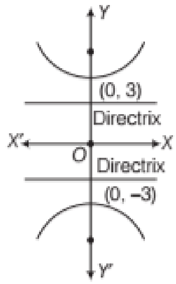
EXAMPLE |1| Draw the shape of the hyperbola $\frac{y^2}{9}-\frac{x^2}{27}=1$ and find their centre, transverse axis, conjugate axis, value of c, vertices, directrices, foci, eccentricity and latusrectum. **[NCERT]**

Sol. Given equation of hyperbola is $\frac{y^2}{9}-\frac{x^2}{27}=1$

On comparing with $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, we get

$$a^2=9, b^2=27 \text{ or } a=3, b=3\sqrt{3}$$

(i) Shape of hyperbola is conjugate hyperbola, because coefficient of x^2 is negative.



(ii) Centre (0, 0)

(iii) Transverse axis, $2a=2\times3=6$

(iv) Conjugate axis, $2b=2\times3\sqrt{3}=6\sqrt{3}$

(v) Value of $c=\sqrt{a^2+b^2}=\sqrt{9+27}=\sqrt{36}=6$

(vi) Vertices $=(0, \pm a)=(0, \pm 3)$

(vii) Directrices, $y=\pm\frac{a^2}{c}=\pm\frac{9}{6}=\pm\frac{3}{2}$

(viii) Foci $=(0, \pm c)=(0, \pm 6)$

(ix) Eccentricity, $e=\frac{c}{a}=\frac{6}{3}=2$

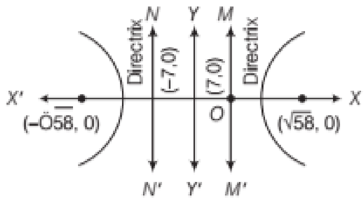
(x) Length of latusrectum $=\frac{2b^2}{a}=\frac{2\times27}{3}=18$

EXAMPLE |2| Draw the shape of the hyperbola $\frac{x^2}{49}-\frac{y^2}{9}=1$ and find their shape, centre, transverse axis, conjugate axis, value of c, vertices, directrices and foci.

Sol. We have, equation of hyperbola $\frac{x^2}{49}-\frac{y^2}{9}=1$

On comparing it with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get $a=7, b=3$

(i) Shape of hyperbola is standard hyperbola, because coefficient of y^2 is negative.



(ii) Centre (0, 0)

(iii) Transverse axis, $2a=2\times7=14$

(iv) Conjugate axis, $2b=2\times3=6$

(v) Value of $c=\sqrt{a^2+b^2}=\sqrt{49+9}=\sqrt{58}$

(vi) Vertices = $(\pm a, 0) = (\pm 7, 0)$

(vii) Directrices, $x = \pm \frac{a^2}{c} = \pm \frac{49}{\sqrt{58}}$

(viii) Foci = $(\pm c, 0) = (\pm \sqrt{58}, 0)$

EXAMPLE |3| Draw the shape of the hyperbola $5y^2 - 9x^2 = 36$ and find its centre, transverse axis, conjugate axis, value of c, vertices, directrices, foci, eccentricity and length of latusrectum.

Sol. We have, equation of hyperbola is $5y^2 - 9x^2 = 36$

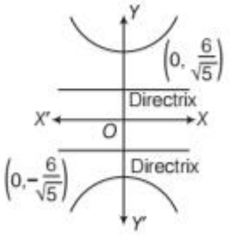
It can be written as

$$\frac{y^2}{\left(\frac{6}{\sqrt{5}}\right)^2} - \frac{x^2}{(2)^2} = 1$$

On comparing with $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we get

$$a^2 = \left(\frac{6}{\sqrt{5}}\right)^2 \text{ and } b^2 = 2^2 \Rightarrow a = \frac{6}{\sqrt{5}} \text{ and } b = 2$$

(i) Shape of hyperbola is conjugate hyperbola, because coefficient of x^2 is negative.



(ii) Centre (0, 0)

(iii) Transverse axis, $2a = 2 \times \frac{6}{\sqrt{5}} = \frac{12}{\sqrt{5}}$

(iv) Conjugate axis, $2b = 2 \times 2 = 4$

(v) Value of $c = \sqrt{a^2 + b^2} = \sqrt{\frac{36}{5} + 4} = \sqrt{\frac{56}{5}} = \frac{2\sqrt{14}}{\sqrt{5}}$

(vi) Vertices = $(0, \pm a) = \left(0, \pm \frac{6}{\sqrt{5}}\right)$

(vii) Directrices, $y = \pm \frac{a^2}{c} = \pm \frac{36 \times \sqrt{5}}{5 \times 2\sqrt{14}} = \pm \frac{18}{\sqrt{70}}$

(viii) Foci = $(0, \pm c) = \left(0, \pm \frac{2\sqrt{14}}{\sqrt{5}}\right)$

(ix) Eccentricity, $e = \frac{c}{a} = \frac{2\sqrt{\frac{14}{5}}}{\frac{6}{\sqrt{5}}} = \frac{\sqrt{14}}{3}$

(x) Length of latusrectum, $\frac{2b^2}{a} = \frac{2 \times 4}{\frac{6}{\sqrt{5}}} = \frac{4\sqrt{5}}{3}$

|TYPE II|

TO FIND THE EQUATION OF HYPERBOLA IN DIFFERENT CASES

When some parts of a hyperbola are given, then for finding the equation of hyperbola, we find the values of a and b with the help of given parts and then put these values in standard equation of hyperbola.

***Case* I When foci and vertices are given**

EXAMPLE |4| Find the equation of the hyperbola, whose vertices are $(0, \pm 5)$ and foci $(0, \pm 8)$. **[NCERT]**

Sol. We have, vertices $(0, \pm a) = (0, \pm 5) \Rightarrow a = 5$

and foci $(0, \pm c) = (0, \pm 8) \Rightarrow c = 8$

We know that, $c^2 = a^2 + b^2 \Rightarrow 64 = 25 + b^2$

$\Rightarrow \quad b^2 = 64 - 25 \Rightarrow b^2 = 39$

Here, the foci and vertices lie on Y-axis, therefore the equation of hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

i.e. $\frac{y^2}{25} - \frac{x^2}{39} = 1$

***Case* II When transverse or conjugate axis and foci are given**

EXAMPLE |5| Find the equation of the hyperbola having foci $(0, \pm 4)$ and transverse axis of length 6.

Sol. We have, the foci of hyperbola lies on Y-axis. So, the equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Length of its transverse axis = $2a$

$\therefore \quad 2a = 6 \Rightarrow a = 3$

Foci $\equiv (0, \pm c) = (0, \pm 4) \Rightarrow c = 4$

We know that, $c^2 = a^2 + b^2$

$\Rightarrow \quad 16 = 9 + b^2 \Rightarrow b^2 = 7$

Thus, $a^2 = 9, b^2 = 7$

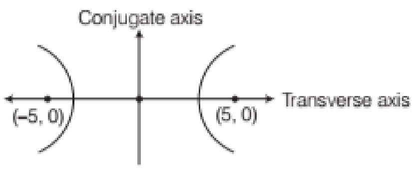
Hence, the equation of hyperbola is $\frac{y^2}{9} - \frac{x^2}{7} = 1$

EXAMPLE |6| Find the equation of hyperbola, when foci are at $(\pm 5, 0)$ and transverse axis is of length 8.

Sol. Here, foci are at $(\pm 5, 0)$. **[NCERT]**

$\therefore \quad (\pm c, 0) = (\pm 5, 0) \Rightarrow c = 5$

and length of transverse axis = $2a = 8 \Rightarrow a = 4$



Also, we know that, $c^2 = a^2 + b^2$

$\Rightarrow \quad 25 = 16 + b^2 \quad [\because a = 4, c = 5]$

$\Rightarrow \quad b^2 = 9$

Since, the foci lie on X-axis. Therefore, the equation of hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

On putting the values of a^2 and b^2, we get

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is the required equation of hyperbola.

EXAMPLE |7| Find the equation of the hyperbola whose foci are at $(0, \pm 6)$ and length of whose conjugate axis is $2\sqrt{11}$.

Sol. We have, foci of the hyperbola lies on Y-axis.

So, the equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Foci $\equiv (0, \pm c) = (0, \pm 6) \Rightarrow c = 6$

Length of its conjugate axis $= 2b$

$\therefore \quad 2b = 2\sqrt{11} \Rightarrow b = \sqrt{11}$

Now, $\quad c^2 = a^2 + b^2 \quad [\because c = 6 \text{ and } b = \sqrt{11}]$

$\Rightarrow \quad 6^2 = a^2 + (\sqrt{11})^2 \Rightarrow a^2 = 36 - 11 = 25$

Thus, $\quad a^2 = 25, b^2 = 11$

Hence, the equation of hyperbola is

$$\frac{y^2}{25} - \frac{x^2}{11} = 1$$

Case III When vertices or foci and eccentricity are given

EXAMPLE |8| Find the equation of the hyperbola whose eccentricity is $\frac{3}{2}$ and foci are $(\pm 2, 0)$.

[NCERT Exemplar]

Sol. We have, foci of the hyperbola lies on X-axis. So, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Foci $\equiv (\pm c, 0) = (\pm 2, 0) \Rightarrow c = 2$

Eccentricity of the hyperbola, $e = \frac{3}{2}$

We know that, $c = ae$

$\therefore \quad 2 = a\left(\frac{3}{2}\right) \Rightarrow a = \frac{4}{3}$

Also, $\quad c^2 = a^2 + b^2$

$\Rightarrow \quad (2)^2 = \left(\frac{4}{3}\right)^2 + b^2 \Rightarrow b^2 = \frac{20}{9}$

Thus, $\quad a^2 = \frac{16}{9}, b^2 = \frac{20}{9}$

Hence, equation of hyperbola is

$$\frac{9x^2}{16} - \frac{9y^2}{20} = 1 \quad \text{[from Eq. (i)]}$$

$\Rightarrow \quad 45x^2 - 36y^2 = 80$

EXAMPLE |9| Find the equation of hyperbola, if vertices are at $(\pm 7, 0)$ and $e = \frac{4}{3}$.

[NCERT]

Sol. We have, vertices of hyperbola lies on X-axis. So, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Vertices $\equiv (\pm a, 0) = (\pm 7, 0) \Rightarrow a = 7$

Now, $\quad e = \sqrt{1 + \frac{b^2}{a^2}} \Rightarrow e^2 = 1 + \frac{b^2}{a^2}$

$\Rightarrow \quad \left(\frac{4}{3}\right)^2 = 1 + \frac{b^2}{7^2} \quad [\because e = 4/3 \text{ and } a = 7]$

$\Rightarrow \quad \frac{b^2}{49} = \frac{16}{9} - 1 \Rightarrow b^2 = \frac{343}{9}$

Thus, $\quad a^2 = 49, b^2 = \frac{343}{9}$

Hence, the equation of hyperbola is

$$\frac{x^2}{49} - \frac{9y^2}{343} = 1 \Rightarrow 7x^2 - 9y^2 = 343 \quad \text{[from Eq. (i)]}$$

Case IV When foci and length of latusrectum are given

EXAMPLE |10| Find the equation of hyperbola whose foci are $(0, \pm 12)$ and the length of latusrectum is 36.

Sol. We have, foci of the hyperbola lies on Y-axis. So, the equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{...(i)}$$

Foci $\equiv (0, \pm c) = (0, \pm 12) \Rightarrow c = 12$

Length of its latusrectum $= \frac{2b^2}{a}$

$\Rightarrow \quad \frac{2b^2}{a} = 36 \Rightarrow b^2 = 18a$

Now, $\quad c^2 = a^2 + b^2 \Rightarrow (12)^2 = a^2 + 18a$

$\Rightarrow \quad a^2 + 18a - 144 = 0 \Rightarrow (a + 24)(a - 6) = 0$

$\Rightarrow \quad a = 6 \quad [\because a = -24 \text{ is neglect}]$

Thus, $a^2 = 36$ and $b^2 = 18a = 18 \times 6 = 108$

Hence, the equation of hyperbola is

$$\frac{y^2}{36} - \frac{x^2}{108} = 1 \Rightarrow 3y^2 - x^2 = 108 \quad \text{[from Eq. (i)]}$$

Case V When vertices and directrix are given

EXAMPLE |11| Find the equation of hyperbola whose vertices are $(\pm 6, 0)$ and one of the directrices is $x = 4$.

Sol. We have, vertices of hyperbola lies on X-axis.

So, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Vertices of hyperbola $\equiv (\pm a, 0) = (\pm 6, 0) \Rightarrow a = 6$

Equation of directrix is $x = \frac{a}{e}$; $\frac{a}{e} = 4$

$$\Rightarrow \quad \frac{6}{e} = 4$$

$$\Rightarrow \quad e = \frac{3}{2}$$

Now, $$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$\Rightarrow \quad a^2e^2 = a^2 + b^2$$

$$\Rightarrow \quad \left(6 \times \frac{3}{2}\right)^2 = 6^2 + b^2$$

$$\Rightarrow \quad b^2 = 9^2 - 6^2 = 45$$

Thus, $a^2 = 36,\ b^2 = 45$

Hence, the equation of hyperbola is $\frac{x^2}{36} - \frac{y^2}{45} = 1$. [from Eq. (i)]

Case VI When focus, equation of directrix and eccentricity are given

If focus $F(x_1, y_1)$, equation of directrix is $ax + by + c = 0$, e is eccentricity and $P(x, y)$ is any point on hyperbola, then equation of hyperbola is

$$(x - x_1)^2 + (y - y_1)^2 = \frac{e^2(ax + by + c)^2}{a^2 + b^2}$$

EXAMPLE |12| Find the equation of the hyperbola whose focus is $(1,1)$, directrix is $2x + y - 1 = 0$ and eccentricity is $\sqrt{3}$.

Sol. Here, focus $F(1, 1)$, equation of directrix is $2x + y - 1 = 0$ and $e = \sqrt{3}$.

Let $P(x, y)$ be any point on the hyperbola, then equation of hyperbola is

$$(x-1)^2 + (y-1)^2 = (\sqrt{3})^2 \cdot \frac{(2x + y - 1)^2}{2^2 + 1^2}$$

$$\Rightarrow \quad 5[(x-1)^2 + (y-1)^2] = 3(2x + y - 1)^2$$

$$\Rightarrow \quad 5(x^2 + y^2 - 2x - 2y + 2)$$

$$= 3(4x^2 + y^2 + 1 + 4xy - 4x - 2y)$$

$$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 4y - 7 = 0$$

$$\Rightarrow \quad 7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$$

Case VII When other parts of hyperbola are given

EXAMPLE |13| Find the equation of hyperbola, the length of whose latusrectum is 8, eccentricity is $\frac{3}{\sqrt{5}}$ and whose transverse and conjugate axes are along the X and Y-axes respectively.

Sol. We have, transverse axis is along the X-axis. So, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Length of its latusrectum $= \frac{2b^2}{a}$

$$\Rightarrow \quad \frac{2b^2}{a} = 8 \Rightarrow b^2 = 4a$$

Now, $$e = \sqrt{1 + \frac{b^2}{a^2}} \Rightarrow \frac{3}{\sqrt{5}} = \sqrt{1 + \frac{4a}{a^2}}$$

$[\because e = 3/\sqrt{5}$ and $b^2 = 4a]$

$$\Rightarrow \quad \frac{9}{5} = 1 + \frac{4}{a} \Rightarrow \frac{4}{a} = \frac{4}{5} \Rightarrow a = 5$$

Thus, $a^2 = 25$ and $b^2 = 4a = 4(5) = 20$

Hence, the equation of hyperbola is $\frac{x^2}{25} - \frac{y^2}{20} = 1$. [from Eq. (i)]

EXAMPLE |14| Find the equation of the locus of all points such that difference of their distance from (4, 0) and (– 4, 0) is always to 2.

Sol. We have, foci of the hydperbola lies on X-axis. So, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Let the given points be F_1 (4, 0) and $F_2(-4, 0)$.

Now, $F_1F_2 = \sqrt{(4+4)^2 + (0-0)^2} = 8$

We know that, $F_1F_2 = 2c$

$\therefore \quad 2c = 8 \Rightarrow c = 4$

Let $P(x, y)$ be any point on the hyperbola.

$\therefore \quad |PF_1 - PF_2| = 2$

Now, $|PF_1 - PF_2| = 2a$

$\therefore \quad 2a = 2 \Rightarrow a = 1$

Also, $c^2 = a^2 + b^2$

$\Rightarrow \quad 16 = 1 + b^2 \Rightarrow b^2 = 15 \quad [\because c = 4$ and $a = 1]$

On putting the values of $a^2 = 1$ and $b^2 = 15$ in Eq. (i), we get

$$\frac{x^2}{1} - \frac{y^2}{15} = 1$$

i.e. $$15x^2 - y^2 = 15$$

which is the required equation of hyperbola.

|TYPE III|

PROBLEMS BASED ON FINDING ECCENTRICITY IN DIFFERENT CONDITIONS

EXAMPLE |15| Find the eccentricity of the hyperbola whose length of latusrectum is 8 and conjugate axis is equal to the half of its distance between the foci.

[NCERT Exemplar]

Sol. We have, conjugate axis is half of distance between foci.

$$\therefore \quad 2b = \frac{1}{2} 2c \Rightarrow 2b = c \Rightarrow 4b^2 = c^2$$

$$\Rightarrow \quad 4b^2 = a^2 + b^2 \Rightarrow \frac{b^2}{a^2} = \frac{1}{3}$$

Now, $e = \sqrt{1 + \frac{b^2}{a^2}}$

$$\Rightarrow \quad e = \sqrt{1 + \frac{1}{3}} \Rightarrow e = \frac{2}{\sqrt{3}}$$

EXAMPLE |16| If e and e' are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate hyperbola, then prove that $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

Sol. Given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The eccentricity e of this hyperbola is

$$e^2 = 1 + \frac{b^2}{a^2} \Rightarrow e^2 = \frac{a^2 + b^2}{a^2}$$

$$\Rightarrow \quad \frac{1}{e^2} = \frac{a^2}{a^2 + b^2} \quad \text{...(i)}$$

The equation of the conjugate hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

The eccentricity e' of this hyperbola is

$$e'^2 = 1 + \frac{a^2}{b^2} \Rightarrow e'^2 = \frac{b^2 + a^2}{b^2}$$

$$\Rightarrow \quad \frac{1}{e'^2} = \frac{b^2}{a^2 + b^2} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1$$

$$\therefore \quad \frac{1}{e^2} + \frac{1}{e'^2} = 1$$

Hence proved.

TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1 Hyperbola is symmetric with respect to

(a) X-axis

(b) Y-axis

(c) X-axis or Y-axis

(d) X-axis and Y-axis

2 If the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$, then

(a) transverse axis is along X-axis of length 6

(b) transverse axis is along Y-axis of length 8

(c) conjugate axis is along Y-axis of length 6

(d) None of the above

3 If the equation of hyperbola is $\frac{y^2}{25} - \frac{x^2}{9} = 1$, then

(a) transverse axis is along X-axis of length 6

(b) transverse axis is along Y-axis of length 10

(c) conjugate axis is along Y-axis of length 10

(d) None of the above

4 If $\frac{x^2}{49} - \frac{y^2}{9} = 1$, the foci of this hyperbola is

(a) (58, 0) (b) $(\pm\sqrt{58}, 1)$

(c) $(\pm\sqrt{58}, 0)$ (d) (58, 1)

5 In a hyperbola, if length of transverse axis is $2a$ and length of conjugate axis is $2b$, then the length of latusrectum is

(a) $\frac{b^2}{a}$ (b) $\frac{2b^2}{a}$

(c) $\frac{a^2}{b}$ (d) $\frac{2a^2}{b}$

VERY SHORT ANSWER Type Questions

6 Find the distance between the directrices of the hyperbola $x^2 - y^2 = 8$.

7 Find the foci of the hyperbola $9x^2 - 16y^2 = 144$.

8 Find the latusrectum of the hyperbola $16x^2 - 9y^2 = 144$.

9 Find the eccentricity of the hyperbola whose length of latusrectum is half of its transverse axis.

SHORT ANSWER Type I Questions

10 If the distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then find the equation of the hyperbola. [NCERT Exemplar]

11 Find the equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13.

12 Find the eccentricity of the hyperbola, the length of whose conjugate axis is $3/4$ of the length of transverse axis.

13 Find the equation of hyperbola having vertices $(\pm 5, 0)$ and foci at $(\pm 7, 0)$.

14 Find the equation of hyperbola having foci $(\pm 5, 0)$ and length of transverse axis is 8. [NCERT]

15 Find the equation of the hyperbola whose vertices are $(\pm 6, 0)$ and one of the directrix is $x = 4$.

16 Find the equation of hyperbola, having foci $(\pm 4, 0)$ and length of latusrectum is 12. [NCERT]

SHORT ANSWER Type II Questions

17 Draw the shape of hyperbola and find the length of the axes, coordinate of the vertices, coordinate of the foci, eccentricity and length of the latusrectum of each of the following hyperbola.

(i) $\frac{y^2}{16} - \frac{x^2}{49} = 1$ (ii) $\frac{x^2}{25} - \frac{y^2}{4} = 1$

(iii) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (iv) $x^2 - y^2 = 1$

(v) $3x^2 - 2y^2 = 6$ (vi) $25x^2 - 9y^2 = 225$

18 Find the foci, vertices, eccentricity and length of latusrectum of the hyperbola $5y^2 - 9x^2 = 36$.

19 Find the lengths of transverse and conjugate axes, eccentricity and coordinate of foci and vertices, length of latusrectum, equation of the directrix of the hyperbola $25x^2 - 36y^2 = 225$.

20 Find the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, when passes through the points $(3, 0)$ and $(3\sqrt{2}, 2)$. [NCERT Exemplar]

21 Find the equation of the hyperbola whose one directrix is $x + y = 9$, the corresponding focus is $(2, 2)$ and eccentricity is 2.

22 Prove that the locus of the point of intersection of the lines $\sqrt{3}x - y = 4\sqrt{3}k$ and $\sqrt{3}kx + ky = 4\sqrt{3}$ for different value of k, is a hyperbola whose eccentricity is 2.

HINTS & ANSWERS

1. (d) Hyperbola is symmetric with respect to both the axes, since if (x, y) is a point on the hyperbola, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the hyperbola.

2. (a) The transverse axis of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is along the X-axis and its length is $2a$.

3. (b) The transverse axis of $\frac{y^2}{a^2} - \frac{y^2}{b^2} = 1$ is along the Y-axis and its length is $2a$.

4. (c) The foci of hyperbola $= (\pm\sqrt{a^2 + b^2}, 0)$
$= (\pm\sqrt{49 + 9}, 0) = (\pm\sqrt{58}, 0)$

5. (b) The length of latusrectum is $\frac{2b^2}{a}$.

6. Distance between directrices $= \frac{2a}{e}$
$a = b = 2\sqrt{2}$ and $e = \sqrt{a^2 + b^2} = 8\sqrt{2}$ **Ans.** 0.5

7. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
Here, $x^2 = 16, b^2 = 9$ and $c = \sqrt{a^2 + b^2}$
$\therefore$ Foci $= (\pm c, 0)$ **Ans.** $(\pm 5, 0)$

8. $\frac{x^2}{9} - \frac{y^2}{16} = 1$
Here, $a^2 = 9,\ b^2 = 16$ and length of latusrectum $= \frac{2b^2}{a}$
Ans. $\frac{32}{3}$

9. $\frac{2b^2}{a} = \frac{1}{2}(2a)$ and use $e = \sqrt{1 + \frac{b^2}{a^2}}$ **Ans.** $\sqrt{\frac{3}{2}}$

10. Here, $2c = 16 \Rightarrow c = 8$
$\because\ e = \frac{c}{a} \Rightarrow \sqrt{2} = \frac{8}{a} \Rightarrow a = 4\sqrt{2}$
$\because\ c^2 = a^2 + b^2$ **Ans.** $x^2 - y^2 = 32$

11. $2b = 5$ and $2c = 13$
$\Rightarrow\ b = \frac{5}{2}$ and $c = \frac{13}{2}$

$\because \quad c^2 = a^2 + b^2 \Rightarrow \frac{169}{4} = \frac{25}{4} + b^2 \Rightarrow a^2 = \frac{12}{2}$

Ans. $\frac{x^2}{144} - \frac{y^2}{25} = \frac{1}{4}$

12. $2b = \frac{3}{4}(2a) \Rightarrow b = \frac{3}{4}a \Rightarrow e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{9}{16}}$ **Ans.** $\frac{5}{4}$

13. Here, $(\pm a, 0) = (\pm 5, 0) \Rightarrow a = 5$

and $(\pm c, 0) = (\pm 7, 0) \Rightarrow c = 7$

$\because \; c^2 = a^2 + b^2 \Rightarrow 49 = 25 + b^2 \Rightarrow b^2 = 24$

Ans. $\frac{x^2}{25} - \frac{y^2}{24} = 1$

14. $(\pm c, 0) = (\pm 5, 0) \Rightarrow c = 5$

and $2a = 8 \Rightarrow a = 4$

$\because \quad c^2 = a^2 + b^2 \Rightarrow 25 = 16 + b^2$

$\Rightarrow \quad b^2 = 9$ **Ans.** $\frac{x^2}{16} - \frac{y^2}{9} = 1$

15. Here $(\pm 6, 0) = (\pm a, 0)$

$\therefore \quad a = 6$ and $x = \frac{a}{e} \Rightarrow 4 = \frac{6}{e} \Rightarrow e = \frac{3}{2}$

$\because \quad b^2 = a^2(e^2 - 1)$ **Ans.** $\frac{x^2}{36} - \frac{y^2}{45} = 1$

16. $\frac{x^2}{4} - \frac{y^2}{12} = 1$

17. (i) Solve as Example-1.

Ans. $8, 14, (0, \pm 4), (0, \pm 65), \frac{\sqrt{65}}{4}, \frac{49}{2}$

(ii) Solve as Example 2.

Ans. $10, 4, (\pm 5, 0), (\pm\sqrt{29}, 0), \frac{\sqrt{29}}{5}, \frac{8}{5}$

(iii) Solve as Example 2. **Ans.** $6, 8, (\pm 3, 0), (\pm 5, 0), \frac{5}{3}, \frac{32}{3}$

(iv) Solve as Example 3. **Ans.** $2, 2, (\pm 1, 0), (\pm\sqrt{2}, 0), \sqrt{2}, 2$

(v) Solve as Example 3.

Ans. $2\sqrt{2}, 2\sqrt{3}, (\pm\sqrt{2}, 0), (\pm\sqrt{5}, 0), \frac{\sqrt{5}}{\sqrt{2}}, 3\sqrt{2}$

(vi) Solve as Example 3.

Ans. $6, 10, (\pm 3, 0), (\pm\sqrt{34}, 0), \frac{\sqrt{34}}{3}, \frac{50}{3}$

18. $\frac{y^2}{\frac{36}{5}} - \frac{x^2}{4} = 1$

Here, $a^2 = \frac{36}{5}$ and $b^2 = 4,\; c = \sqrt{a^2 + b^2}$

$= \sqrt{\frac{36}{5} + 4} = \sqrt{\frac{56}{5}} = 2\sqrt{\frac{14}{5}}$

Ans. $\left(0, \pm\frac{2\sqrt{14}}{\sqrt{5}}\right), \left(0, \pm\frac{6}{\sqrt{5}}\right), \frac{\sqrt{14}}{3}, \frac{4\sqrt{5}}{3}$

19. $\frac{x^2}{9} - \frac{y^2}{\frac{225}{36}} = 1$

Here, $a^2 = 9,\; b^2 = \frac{225}{36}$

and $c^2 = \sqrt{a^2 + b^2}$

$= \sqrt{9 + \frac{225}{36}}$

$= \sqrt{\frac{549}{36}} = \frac{\sqrt{549}}{6}$

$= \frac{3\sqrt{61}}{6} = \frac{\sqrt{61}}{2}$

Ans. $6, 5, \frac{\sqrt{61}}{6}, \left(\pm\frac{\sqrt{61}}{2}, 0\right), (\pm 3, 0), \frac{25}{6}, x = \pm\frac{18}{\sqrt{61}}$

20. Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Since, it is passes through (3, 0) and $(3\sqrt{2}, 2)$.

$\therefore \quad \frac{9}{a^2} - 0 = 1$ and $\frac{18}{a^2} - \frac{4}{b^2} = 1$

$\Rightarrow \quad a^2 = 9$ and $b^2 = 4$

$\because \quad b^2 = a^2(e^2 - 1)$ **Ans.** $e = \frac{\sqrt{13}}{3}$

21. $(x-2)^2 + (y-2)^2 = \frac{2^2(x+y-9)^2}{(1^2+1^2)}$

$\Rightarrow x^2 + y^2 - 4x - 4y + 8 = 2(x^2 + y^2 + 81 + 2xy - 18y - 18x)$

Ans. $x^2 + 4xy + y^2 - 32x - 32y + 154 = 0$

SUMMARY

- ,A **circle** is the locus of a point in a plane, which moves in a plane such that its distance from a fixed point in that plane is always constant.
- **Standard Equation** of a circle is $(x-h)^2+(y-k)^2=r^2$ having centre (h, k) and radius r.
- The **General Equation** of a circle is $x^2+y^2+2gx+2fy+c=0$ having **centre** $(-g, -f)$ and radius $r=\sqrt{g^2+f^2-c}$.
- Let (x_1, y_1) and (x_2, y_2) be the end points of the diameter of a circle. Then, equation of circle is
$$(x-x_1)\cdot(x-x_2)+(y-y_1)\cdot(y-y_2)=0$$
- A **parabola** is the locus of a point which moves in a plane, so that its distance from a fixed point called **focus** is always equal to its distance from a fixed straight line called **directrix** in the same plane.
- Let the equation of a parabola be $y^2=4ax$.
  (i) The focus and directrix of a parabola be $(a, 0)$ and $x=-a$. (ii) The length of the latusrectum is $4a$.
- An **ellipse** is the locus of a point in a plane, which moves in the plane such that the ratio of its distance from a fixed point called **focus** in the same plane to its distance from a fixed straight line called **directrix** is always constant, which is always less than unity.
  (i) The standard equation of **horizontal ellipse** is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$.
  (ii) The standard equation of **vertical ellipse** is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, a>b$.
- Let the equation of an **ellipse** be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$.
  (i) The major and minor axes are $2a$ and $2b$, respectively.
  (ii) The foci of an ellipse is $(\pm c, 0)$, where $c=\sqrt{a^2-b^2}$. (iii) The eccentricity of an ellipse is $e=\frac{c}{a}=\sqrt{1-\frac{b^2}{a^2}}$.
  (iv) The directrices of an ellipse are $x=\pm\frac{a^2}{c}$, where $c=\sqrt{a^2-b^2}$. (v) The length of latusrectum is $\frac{2b^2}{a}$.
- A **hyperbola** is the locus of a point in a plane, which moves in the plane such that the ratio of its distances from a fixed point called **focus** in the same plane to its distance from a fixed line called **directrix** is always constant, which is always greater than unity.
  (i) The standard equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
  (ii) The equation of the conjugate hyperbola is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.
- Let equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
  (i) The transverse and conjugate axes are $2a$ and $2b$, respectively.
  (ii) The foci of hyperbola is $(\pm c, 0)$ or $(\pm ae, 0)$.
  (iii) The directrices of a hyperbola are $x=\pm\frac{a^2}{c}$ or $\pm\frac{a}{e}$, where $c=\sqrt{a^2+b^2}$ and $e=\sqrt{1+\frac{b^2}{a^2}}$
  (iv) The length of latusrectum is $\frac{2b^2}{a}$.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1. The vertex separates the double napped right circular cone into ...M... parts called ...N... . Here, M and N respectively stand for

(a) three, nappes (b) two, shapes
(c) two, nappes (d) three, cones

2. Different kinds of conic sections are obtained depending on

(a) the position of the intersecting plane with respect to the cone
(b) angle made by intersecting plane with the vertical axis of the cone
(c) Both (a) and (b)
(d) Neither (a) nor (b)

3. A circle is the set of all points in a plane that are equidistant from a ...P... point in the ...Q... . Here, P and Q respectively are

(a) any, space (b) fixed, space
(c) any, plane (d) fixed, plane

4. If the equation of the circle with centre at (h, k) and radius r is $x^2 + y^2 = r^2$. Then, h and k respectively are

(a) (1, 1) (b) (−1, −1)
(c) (0, 0) (d) None of these

5. Vertex of the parabola $9x^2 - 6x + 36y + 9 = 0$, is

(a) $\left(\frac{1}{3}, -\frac{2}{9}\right)$ (b) $\left(-\frac{1}{3}, -\frac{1}{2}\right)$
(c) $\left(-\frac{1}{3}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{3}, \frac{1}{2}\right)$

6. The length of the latusrectum of the parabola $9x^2 - 6x + 36y + 19 = 0$, is

(a) 36 (b) 9 (c) 6 (d) 4

7. Latusrectum of an ellipse is a line segment ...A... to the major axis through any of the ...B... and whose end points lie on the ellipse. Here, A and B respectively are

(a) perpendicular, foci (b) parallel, vertices
(c) parallel, foci (d) perpendicular, vertices

8. The equation $\frac{x^2}{2-r} + \frac{y^2}{r-5} + 1 = 0$ represents an ellipse, if

(a) $r > 2$
(b) $2 < r < 5$
(c) $r > 5$
(d) None of the above

9. The number of possible orientations of hyperbola is/are

(a) one (b) two
(c) three (d) four

10. A hyperbola in which length of transverse axis is equal to the length of conjugate axis, is called

(a) equilateral hyperbola
(b) obtuse hyperbola
(c) acute hyperbola
(d) None of the above

VERY SHORT ANSWER Type Questions

11. Find the equation of the parabola whose focus is (2, 0) and directrix is $x = -2$.

12. Find the equation of ellipse having ends of major and minor axes are $(0, \pm\sqrt{5})$ and $(\pm 1, 0)$. [NCERT]

13. Find the equation of the ellipse, having eccentricity $\frac{1}{2}$ and semi-major axis is 4.

14. Find the equation of hyperbola, if length of transverse axis is 10 and conjugate axis is 8.

SHORT ANSWER Type I Questions

15. Find the centre and radius of each of the following circles. (Each part carries 2 marks)

(i) $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{4}$

(ii) $x^2 + y^2 - 4x + 6y = 5$

16. Find the coordinates of the focus and the vertex, the equation of directrix and the axis of the following (Each part carries 2 Marks)

(i) $x^2 = 10y$ (ii) $3x^2 = 8y$

17. Find the coordinates of the focus and the vertex, equation of directrix and length of latusrectum of the following.
(Each part carries 2 marks)

(i) $y^2 = 10x$ (ii) $5y^2 = -16x$

18. Find the equation of the parabola with vertex at the origin, the axis along the X-axis and passing through the point (2, 3).

19. Find the equation of the parabola whose focus is (–1, 2) and directrix is $x - 2y - 15 = 0$.

20. Find the distance between the directrices of the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$.

21. Find the equation of an ellipse whose vertices are (0, ±10) and eccentricity $e = \frac{4}{5}$.

22. Find the equation of the hyperbola with vertices at (0, ± 6) and $e = \frac{5}{3}$. Find its foci. [NCERT Exemplar]

23. Find the equation of the hyperbola, having foci $(\pm 3\sqrt{5}, 0)$ and length of latusrectum is 8.

24. Find the equation of the hyperbola whose vertices are (± 2, 0) and the eccentricity is 2.

25. Find the equation of hyperbola, if conjugate axis is 5 and distance between foci is 13.

SHORT ANSWER Type II Questions

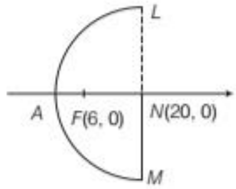
26. If the equations of the two diameters of a circle are $x + y = 6$ and $x + 2y = 4$ and the radius of the circle is 10. Find the equation of the circle.

27. If two diameters of a circle lie along the lines $x - y = 9$ and $x - 2y = 7$ and the area of the circle is 38.5 sq cm, then find the equation of the circle.

28. Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ and which touches the Y-axis.

29. Find the focus and directrix of the parabola $3x^2 + 12x + 8y = 0$.

30. The focus of a parabolic mirror as shown in figure is at a distance 6 cm from its vertex. If the mirror is 20 cm deep, then find the distance LM.



31. Find the equation of the ellipse, whose focus is (1, – 2), the directrix $3x - 2y + 5 = 0$ and eccentricity equal to $\frac{1}{2}$.

32. If the latusrectum of an ellipse with axis along X-axis and centre at origin is 10, distance between foci is equal to length of minor axis, then find the equation of ellipse. [NCERT]

33. PSQ is a focal chord of the ellipse $4x^2 + 9y^2 = 36$ such that $SP = 4$. If S' is the another focus, write the value of $S'Q$.

34. Find the coordinates of the foci, vertices, eccentricity and length of latusrectum of the following hyperbola. (Each part carries 4 Marks)

(i) $49y^2 - 16x^2 = 784$ (ii) $16x^2 - 9y^2 = 576$

35. Find the length of the axes, vertices, foci, eccentricity and length of the latusrectum of the following hyperbola. (Each part carries 4 Marks)

(i) $9x^2 - 16y^2 = 144$ (ii) $9y^2 - 4x^2 = 36$

36. Find the equation of hyperbola, having directrix $x - y + 3 = 0$, focus (–1, 1) and eccentricity 3.

LONG ANSWER Type Questions

37. The sides of a rectangle are given by the equations $x = -2$, $x = 4$, $y = -2$ and $y = 5$. Find the equation of the circle drawn on the diagonal of this rectangle as its diameter.

38. Find the axis, vertex, directrix and length of latusrectum of the parabola $9y^2 - 16x - 12y - 57 = 0$.

39. Find the equation of an ellipse whose axis lie along the coordinate axes, which passes through the point (–3, 1) and has eccentricity equal to $\sqrt{2/5}$.

40. Find the equation of the hyperbola whose foci are at $(0, \pm\sqrt{10})$ and which passes through the point (2, 3).

HINTS & ANSWERS

1. (c) The vertex separates the cone into two parts called nappes.

2. (c) We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and the angle made by intersecting plane with the vertical axis of the cone.

3. (d) A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.

4. (c) Suppose $h = k = 0$. Therefore, the equation of the circle is $x^2 + y^2 = r^2$.

5. (a) The given equation $9x^2 - 6x + 36y + 9 = 0$ can be written as $(3x-1)^2 = -4(9y+2)$.

Hence, the vertex is $\left(\frac{1}{3}, -\frac{2}{9}\right)$.

6. (d) $9x^2 - 6x + 19 = -36y$

$$\Rightarrow \quad (3x-1)^2 = -36y - 18 = -36\left(y + \frac{1}{2}\right)$$

$$\Rightarrow \quad 9\left(x - \frac{1}{3}\right)^2 = -36\left(y + \frac{1}{2}\right)$$

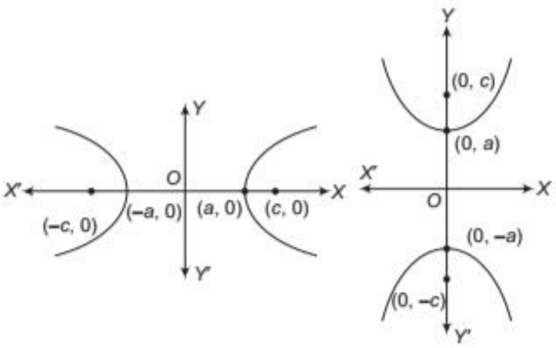
Hence, length of latusrectum is 4.

7. (a) Latusrectum of an ellipse is a line segment **perpendicular** to the major axis through any of the **foci** and whose end points lie on the ellipse.

8. (b) $\frac{x^2}{2-r} + \frac{y^2}{r-5} + 1 = 0 \Rightarrow \frac{x^2}{r-2} + \frac{y^2}{5-r} = 1$

Hence, $r > 2$ and $r < 5 \Rightarrow 2 < r < 5$.

9. (b) The number of possible orientations of hyperbola are two which are shown below.



10. (a) If length of transverse axis is equal to the length of conjugate axis, then hyperbola is called equilateral hyperbola.

11. $\because$ Directrix is $x = -2$ and the focus is $(2, 0)$.

$\therefore$ Required equation of parabola is $y^2 = 4ax$

$$\Rightarrow \quad y^2 = 4 \times 2x \quad [\because a = 2]$$

Ans. $y^2 = 8x$

12. Since, major axis $(0, \pm\sqrt{5})$ is along Y-axis and minor axis $(\pm 1, 0)$ is along X-axis.

$\therefore$ Equation of the ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ ...(i)

where, $a = \sqrt{5}$ and $b = 1$

On putting $a = \sqrt{5}$ and $b = 1$ in Eq. (i), we get

$$\frac{x^2}{1} + \frac{y^2}{5} = 1$$

13. Given, $e = \frac{1}{2}$ and $a = 4$

$$\therefore \quad e^2 = \frac{a^2 - b^2}{a^2} \Rightarrow \frac{1}{4} = \frac{16 - b^2}{16} \Rightarrow b^2 = 12$$

$\therefore$ Equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{12} = 1$ **Ans.** $3x^2 + 4y^2 = 48$

14. $\because$ Transverse axis $= 2a = 10 \Rightarrow a = 5$

and conjugate axis $= 2b = 8 \Rightarrow b = 4$

$\therefore$ Equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ i.e. } \frac{x^2}{25} - \frac{y^2}{16} = 1$$

15. (i) On comparing the given equation with $(x-h)^2 + (y-k)^2 = r^2$, we get

$$h = \frac{1}{2}, k = -\frac{1}{3}$$

and $r = \frac{1}{2}$ **Ans.** $\left(\frac{1}{2}, -\frac{1}{3}\right)$ and $\frac{1}{2}$

(ii) On comparing the given equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$g = -2, f = 3 \text{ and } c = -5$$

Now, centre $\equiv (-g, -f)$

and radius $= \sqrt{g^2 + f^2 - c}$

Ans. $(2, -3)$ and $3\sqrt{2}$

16. (i) $F\left(0, \frac{5}{2}\right)$; $O(0, 0)$; $2y + 5 = 0$; $x = 0$

(ii) $F\left(0, \frac{2}{3}\right)$, $O(0, 0)$; $3y + 2 = 0$; $x = 0$

17. (i) $F\left(\frac{5}{2}, 0\right)$; $O(0, 0)$; $2x + 5 = 0$; 10 units

(ii) $F\left(-\frac{4}{5}, 0\right)$; $O(0, 0)$; $5X - 4 = 0$; $\frac{16}{5}$ units

18. Equation of the parabola with vertex at the origin, the axis along the X-axis is

$$y^2 = 4ax \quad \text{...(i)}$$

Since, parabola (i) passes through (2, 3).

$\therefore \quad 9 = 4 \times a \times 2$ [from Eq. (i)]

$\Rightarrow \quad 9 = 8a \Rightarrow a = \frac{9}{8}$

On putting $a = \frac{9}{8}$ in Eq. (i), we get

$$y^2 = 4 \times \frac{9}{8}x = \frac{9}{2}x$$

Hence, the required equation of parabola is $y^2 = \frac{9}{2}x$.

19. Let $P(x, y)$ be any point on the parabola. Distance of $P(x, y)$ from the focus $(-1, 2)$ = Distance of $P(x, y)$ from the directrix $x - 2y - 15 = 0$ [by definition of parabola]

$\therefore$ Required equation of the parabola is

$$(x+1)^2 + (y-2)^2 = \left|\frac{x - 2y - 15}{\sqrt{1^2 + (-2)^2}}\right|^2$$

Ans. $4x^2 + 4xy + y^2 + 40x - 80y - 200 = 0$

20. Given, $\frac{x^2}{36} + \frac{y^2}{20} = 1$

On comparing the above equation with

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$a^2 = 36 \Rightarrow a = 6 \text{ and } b^2 = 20 \Rightarrow b = 2\sqrt{5}$$

$\because \quad e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{36 - 20}}{6} = \frac{4}{6} = \frac{2}{3}$

$\therefore$ Distance between the directrices $= \frac{2a}{e} = \frac{2 \times 6}{2/3} = 18$

21. Given, vertices $(0, \pm 10) = (0, \pm a)$ and $e = \frac{4}{5}$

$\therefore \quad a = 10$

Now, $\quad e^2 = \left(\frac{4}{5}\right)^2$

$\Rightarrow \quad 1 - \frac{b^2}{a^2} = \frac{16}{25} = \frac{b^2}{100} = 1 - \frac{6}{25} \Rightarrow b^2 = 36$

$\therefore$ Required equation of an ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \Rightarrow \frac{x^2}{36} + \frac{y^2}{100} = 1$$

Ans. $100x^2 + 36y^2 = 3600$

22. Let vertices $\equiv (0, \pm b) = (0, \pm 6)$

$$b = 6 \text{ and } e = \frac{5}{3}$$

$\therefore \quad e^2 = \frac{a^2 + b^2}{b^2} \Rightarrow \frac{25}{9} = \frac{36 + a^2}{36} \Rightarrow a^2 = 48$

Hence, the required equation of hyperbola is

$-\frac{x^2}{48} + \frac{y^2}{36} = 1 \Rightarrow \frac{y^2}{36} - \frac{x^2}{48} = 1$

$\therefore \quad \text{foci} = (0, \pm be) = (0, \pm 10)$

23. Let foci $(\pm c, 0) \equiv (\pm 3\sqrt{5}, 0) \Rightarrow c = 3\sqrt{5}$

and length of latusrectum $= \frac{2b^2}{a} = 8 \Rightarrow b^2 = 4a$

On putting $c = 3\sqrt{5}$ and $b^2 = 4a$ in $c^2 = a^2 + b^2$, we get

$$45 = a^2 + 4a \Rightarrow a^2 + 4a - 45 = 0$$

$\Rightarrow \quad a = -9$ or $a = 5$

But $\quad a \neq -9 \Rightarrow a = 5$

On putting $a = 5$ in $b^2 = 4a$, we get $b^2 = 20$

Hence, the required equation of hyperbola is

$$\frac{x^2}{25} - \frac{y^2}{20} = 1.$$

24. Let vertices $(\pm a, 0) \equiv (\pm 2, 0)$

$\Rightarrow a = 2$ and eccentricity, $e = 2$

On putting $a = 2$ and $e = 2$ in $e^2 = \frac{a^2 + b^2}{a^2}$, we get

$$4 = \frac{4 + b^2}{4} \Rightarrow b^2 = 12$$

Hence, the required equation of hyperbola is

$\frac{x^2}{4} - \frac{y^2}{12} = 0.$

25. Let conjugate axis, $2b = 5 \Rightarrow b = \frac{5}{2}$

and distance between foci $= 2c = 13 \Rightarrow c = \frac{13}{2}$

On putting $b = \frac{5}{2}$ and $c = \frac{13}{2}$ in $c^2 = a^2 + b^2$, we get

$$\frac{169}{4} = a^2 + \frac{25}{4} \Rightarrow a^2 = \frac{144}{4}$$

Hence, the required equation of hyperbola is

$$\frac{x^2}{144/4} - \frac{y^2}{25/4} = 1 \Rightarrow 25x^2 - 144y^2 = 900$$

26. Given, radius, $r = 10$ and equations of two diameters are

$$x + 2y = 4 \quad \text{...(i)}$$

and $\quad x + y = 6 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x = 8 \text{ and } y = -2$$

So, centre of the circle is $(8, -2)$.

$\therefore$ Required equation of circle is

$$(x-8)^2 + (y+2)^2 = (10)^2.$$

Ans. $x^2 + y^2 - 16x + 4y - 32 = 0$

27. Given equations of two diameters are

$$x - y = 9 \quad \text{...(i)}$$

and $\quad x - 2y = 7 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$x = 11 \text{ and } y = 2$$

$\because$ Area of given circle $= 38.5 \text{ cm}^2$

$\Rightarrow \quad \pi r^2 = 38.5 \Rightarrow r^2 = \frac{1225}{100} = \frac{49}{4}$

$\therefore$ Required equation circle with centre (11, 2) and radius $= \frac{7}{2}$ is $(x-11)^2 + (y-2)^2 = \frac{49}{4}$.

Ans. $4x^2 + 4y^2 - 88x - 16y + 451 = 0$

28. Given, $x^2 + y^2 - 4x - 6y - 3 = 0$

$\Rightarrow (x^2 - 4x + 4) + (y^2 - 6y + 9) = 3 + 4 + 9$

$\Rightarrow (x-2)^2 + (y-3)^2 = 16 \Rightarrow (x-2)^2 + (y-3)^2 = (4)^2$

$\therefore$ Centre of the required circle is (2, 3).

As it touches Y-axis, so its radius = x-coordinate of centre = 2.

Hence, the required circle is $(x-2)^2 + (y-3)^2 = (2)^2$

$\Rightarrow x^2 - 4x + 4 + y^2 - 6y + 9 = 4$

$\Rightarrow x^2 + y^2 - 4x - 6y + 9 = 0$

29. Given, $3x^2 + 12x + 8y = 0$

$\Rightarrow x^2 + 4x + \frac{8}{3}y = 0$ [dividing both sides]

$\Rightarrow x^2 + 4x + 4 = -\frac{8}{3}y + 4$

$\Rightarrow (x+2)^2 = -4\left(\frac{2}{3}y - 1\right)$

Let $X = x + 2$

and $Y = \left(\frac{2}{3}y - 1\right)$ ...(i)

$\therefore X^2 = -4Y$ ...(ii)

Coordinates of focus of a parabola (ii) are

$X = 0$ and $Y = -1$

$\Rightarrow x + 2 = 0$ and $\frac{2}{3}y - 1 = -1 \Rightarrow x = -2$ and $y = 0$

$\therefore$ Equation of directrix of parabola (ii) is $Y = -1$

i.e. $\frac{2}{3}y - 1 = 1 \Rightarrow \frac{2}{3}y = 2 \Rightarrow y = 3$

Ans. Focus $\equiv (-2, 0)$ and directrix, $y = 3$

30. $8\sqrt{30}$ cm

31. Given, focus $(1, -2)$ and equation of directrix

$3x - 2y + 5 = 0$ and eccentricity, $e = \frac{1}{2}$.

$\therefore$ The equation of ellipse is

$$(x-1)^2 + (y+2)^2 = \left(\frac{1}{2}\right)^2 \frac{(3x - 2y + 5)^2}{3^2 + (-2)^2}$$

$\Rightarrow x^2 + 1 - 2x + y^2 + 4 + 4y$

$$= \frac{1}{4}\left[\frac{9x^2 + 4y^2 + 25 + 30x - 12xy - 20y}{9+4}\right]$$

$\Rightarrow x^2 + y^2 - 2x + 4y = \frac{1}{52}$

$[9ax^2 + 4y^2 - 12xy + 30x - 20y + 25]$

$\Rightarrow 52x^2 + 52y^2 - 104x + 208y + 260$

$= 9x^2 + 4y^2 - 12xy + 30x - 20y + 25$

$\Rightarrow 43x^2 + 48y^2 + 12xy - 134x + 228y + 235 = 0$

32. Given, $\frac{2b^2}{a} = 10$

and $2ae = 2b \Rightarrow b = ae \because b^2 = a^2(1 - e^2)$

$\Rightarrow e = \frac{1}{\sqrt{2}}$ [using $b = ae$]

Thus, $a = b\sqrt{2}$

Again, $\frac{2b^2}{a} = 10 \Rightarrow b = 5\sqrt{10}$ [$\because a = 10$]

Hence, required equation of the ellipse is $\frac{x^2}{100} + \frac{y^2}{50} = 1$.

33. $\frac{26}{5}$

34. (i) Given, $49y^2 - 16x^2 = 784$

$\Rightarrow \frac{y^2}{16} - \frac{x^2}{49} = 1$ [dividing both sides by 784]

On comparing it with $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we get

$a = 4$ and $b = 7$

Now, transverse axis, $2a = 8$

Conjugate axis, $2b = 14$

Value of $c = \sqrt{a^2 + b^2} = \sqrt{16 + 49} = \sqrt{65}$

Vertices $\equiv (0, \pm a) = (0, \pm 4)$

Foci $\equiv (0, \pm c) = (0, +\sqrt{65})$

Eccentricity $= e = \frac{c}{a} = \frac{\sqrt{65}}{4}$

and length of latusrectum

$= \frac{2b^2}{a} = \frac{2 \times 49}{4} = \frac{49}{2}$

(ii) Solve as part (i).

Ans. Foci $\equiv (\pm 10, 0)$

Vertices $\equiv (\pm 6, 0)$

Eccentricity $= e = \frac{5}{3}$

Length of latusrectum $= \frac{64}{3}$

35. (i) Given, $9x^2 - 16y^2 = 144$

$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$ [dividing both sides by 144]

On comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a = 4$ and $6 = 3$

Now, transverse axis, $2a = 8$

Conjugate axis, $2b = 6$

Value of $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$

Vertices $\equiv (\pm a, 0) = (\pm 4, 0)$,

Foci $\equiv (\pm c, 0) = (\pm 5, 0)$

Eccentricity $= e = \frac{c}{a} = \frac{5}{4}$.

and length of latusrectum $= \frac{2b^2}{a} = \frac{2 \times 3^2}{4} = \frac{9}{2}$

(ii) Solve as part (i).

Ans. Transeverse axis = 4

Conjugate axis = 6

Vertices $\equiv (0, \pm 2)$

Foci $\equiv (0, \pm \sqrt{3})$

Eccentricity $= \frac{\sqrt{13}}{2}$

Length of latusrectum = 9

36. Given, focus $= (-1, 1)$, equation of directrix is

$$x - y + 3 = 0$$

and Eccentricity, $e = 2$

$$\because (x - x_1)^2 + (y - y_1)^2 = \frac{e^2 (ax + by + c)^2}{a^2 + b^2}$$

$$\therefore (x + 1)^2 + (y - 1)^2 = \frac{3(x - y + 3)^2}{1^2 + (-1)^2}$$

Ans. $7(x^2 + y^2) - 18xy + 50x - 50y + 77 = 0$

37. $x^2 + y^2 - 2x - 3y - 18 = 0$

38. Given equation of parabola is

$$9y^2 - 16x - 12y - 57 = 0$$

$$\Rightarrow (3y)^2 - 2 \times 2 \times 3y + 2^2 = 2^2 + 57 + 16x$$

$$\Rightarrow (3y - 2)^2 = 61 + 16x \Rightarrow 9\left(y - \frac{2}{3}\right)^2 = 16\left(x + \frac{61}{16}\right)$$

$$\Rightarrow \left(y - \frac{2}{3}\right)^2 = \frac{16}{9}\left(x + \frac{61}{16}\right) \quad \text{...(i)}$$

Let $Y = y - \frac{2}{3}$ and $X = x + \frac{61}{16}$ ...(ii)

$\therefore$ $Y^2 - \frac{16}{9} X^2$ [from Eqs. (ii) and iii)] ...(iii)

Now, the equation of axis of parabola (iii) is $Y = 0$

$$\therefore y - \frac{2}{3} = 0 \Rightarrow y = \frac{2}{3}$$

Coordinates of vertices of parabola (iii) are $X = 0$ and $Y = 0$

i.e. $y - \frac{2}{3} = 0 \Rightarrow x = -\frac{61}{16}$ and $y = \frac{2}{3}$

On comparing Eq. (iii) with $y^2 = 4ax$, we get

$$4a = \frac{16}{6} \Rightarrow a = \frac{4}{9}$$

$\therefore$ equation of directrix of parabola (iii) is $x = -a$

i.e. $x + \frac{61}{16} = -\frac{4}{9} \Rightarrow x = -\frac{613}{144}$

and length of latusrectum $= |4a| = \left|4 \times \frac{4}{9}\right| = \frac{16}{9}$

39. Let equation of ellipse is, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ ... (i)

and eccentricity, $e = \sqrt{\frac{2}{5}} \Rightarrow e^2 = \frac{2}{5}$

According to the question, ellipse (i) passes through the point $(-3, 1)$.

$\therefore$ $\frac{9}{a^2} + \frac{1}{b^2} = 1$ [from Eq. (i)] ...(ii)

$\because b^2 = a^2(1 - e^2) \therefore b^2 = a^2\left(1 - \frac{2}{5}\right) \Rightarrow b^2 = \frac{3}{5}a^2$ ...(ii)

On substituting $b^2 = \frac{3}{5}a^2$ in Eq. (ii), we get

$$\frac{9}{a^2} + \frac{5}{3a^2} = 1 \Rightarrow 3a^2 = 32 \Rightarrow a^2 = \frac{32}{3}$$

Now, $b^2 = \frac{3}{5} \times \frac{32}{3} = \frac{32}{5}$ [from Eq. (iii)]

On substituting $a^2 = \frac{32}{3}$ and $b^2 = \frac{32}{5}$ in Eq. (i), we get

$$\frac{x^2}{32/3} + \frac{y^2}{32/5} = 1 \Rightarrow 3x^2 + 5y^2 = 32$$

40. Given, foci $= (0, \pm\sqrt{10}) \equiv (0, \pm c)$ (say)

$$\Rightarrow c = \sqrt{10}$$

$$\because c^2 = a^2 + b^2 \Rightarrow 10 = a^2 + b^2$$

$$\Rightarrow a^2 = 10 - b^2 \quad \text{...(i)}$$

Let the required equation of hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{...(ii)}$$

According to the question, this hyperbola passes through (2, 3).

$\therefore$ $\frac{9}{a^2} - \frac{4}{b^2} = 1$ [from Eq. (ii)]

$\Rightarrow$ $\frac{9}{10 - b^2} - \frac{4}{b^2} = 1$ [from Eq. (i)]

$$\Rightarrow \frac{9b^2 - 40 + 4b^2}{(10 - b^2)b^2} = 1$$

$$\Rightarrow 13b^2 - 40 = 10b^2 - b^4$$

$$\Rightarrow b^4 + 3b^2 - 40 = 0$$

$$\Rightarrow b^2 = 5 \text{ or } -8$$

But $b^2 > 0 \Rightarrow b^2 = 5$

On substituting $b^2 = 5$ in Eq. (i), we get

$$a^2 = 10 - 5 = 5$$

On substituting $a^2 = 5$ and $b^2 = 5$ in Eq.(ii), we get,

$$\frac{y^2}{5} - \frac{x^2}{5} = 1$$

$$\Rightarrow y^2 - x^2 = 5$$